

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SEVENTH SERIES.]

JUNE 1935.

LXXXIX. *A new Method of Evaluation of the Heaviside Operational Expression by Fourier Series.* By S. KOIZUMI, Assistant Professor of the Waseda University, Tokyo*.

§ 1. *Introduction.*

THE Heaviside operational calculus is used extensively to solve differential equations. On the other hand, a Volterra integral equation whose nucleus is a function of $(x^n - \xi^n)$ can be treated also operationally, since the equation is reducible, by means of changes of variables, to that considered in a previous paper †.

It is a comparatively simple matter to solve differential equations, integral equations, or integro-differential equations operationally; nevertheless, it often happens that the evaluation of the operational solution there obtained is too difficult to know the behaviour of its explicit function.

* Communicated by the Author.

† Phil. Mag. xi. p. 432 (1931). We take the opportunity of noting the following corrections: p. 436, line 17, for $f'(o)$ read $f(o)$; p. 437, line 22, for $\gamma(x)$ read $f(x)$.

As for the evaluation of operational expressions, following methods are adopted in general :—

1. Series evaluation.
2. Expansion theorem evaluation.
3. Contour integral evaluation.
4. Integral table evaluation.

The procedure of evaluation expanding the operational expression in a power series of the operator is the most direct one and only the simplest sort of mathematics is involved. But the series, whether convergent or asymptotic, is, as a rule, only convenient to compute the value of the explicit function for some very restricted range of the independent variable.

By virtue of the expansion theorem we can obtain physically interesting explicit solutions. But in practical evaluations we often encounter a fatal difficulty in finding the roots of a higher-degree algebraic equation or a transcendental one. In the latter case there are usually infinite numbers of roots, the solution becoming an infinite series, which converges very slowly in general.

The contour integral solution is a very compact one and is convenient for theoretical investigations, but there is a practical difficulty in finding poles and their residues of the integrand.

To find the explicit solution by integral tables is the simplest method provided that we can use an integral formula by slight modifications, otherwise we must adopt another method.

Thus, every method has its merits and demerits, so we must select one according to the problem. There are, however, many operational expressions which cannot be evaluated in a practical way by any method mentioned above, and a need of another method of attack is often felt.

To meet this need to a certain extent a new method of evaluation is proposed in this paper, where the explicit function for the independent variable between zero and s , taken at one's pleasure, is obtained as a Fourier series whose coefficients are calculated in the mere algebraic way from the given operational expression.

According to this method the operational expressions which are encountered in many physical and technical

problems may be evaluated in a simpler manner than heretofore, and the author hopes that his method, associated with others mentioned above, will play an important part in extending the class of problems which can be treated operationally.

§ 2. Calculation of Fourier Coefficients from the Operational Expression.

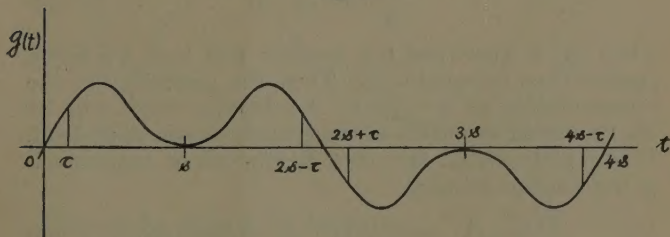
Let $g(t)$ be a function with an absolutely convergent integral and with limited total fluctuation in the range $0 \leq t \leq s$ such that

$$g(0)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$g(\tau)=g(2s-\tau)=-g(2s+\tau)=-g(4s-\tau), \quad . \quad . \quad (2)$$

as shown in fig. 1.

Fig. 1.



Then this function will be represented in a sine series as follows :

$$g(t)=\sum_n b_n \sin vt, \quad (n=1, 3, 5 \dots), \quad . \quad . \quad (3)$$

where

$$v=\frac{\pi n}{2s}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$b_n=\frac{2}{s} \int_0^s g(t) \sin vt \, dt. \quad . \quad . \quad . \quad . \quad (5)$$

In case $g(t)$ is discontinuous at t , this series represents $\{g(t-o)+g(t+o)\}/2$.

Let us now assume that the product of $e^{-\lambda t}$ and a function $f(t)$, whose operational expression is $F(p)$, can be represented by a Fourier series in the interval $0 \leq t \leq s$, where

s may be any positive finite number and λ is another positive number which must be selected by further consideration.

Since the Fourier integral representation of a function in the range $(0, \infty)$ may be regarded as the limiting form of the Fourier series for the function in the interval $0 \leq t \leq s$ when s increases without limit, if a function can be represented by a Fourier integral, it will also be possible to represent the function in a Fourier series in any interval $0 \leq t \leq s$.

On the other hand, it will be proved that $e^{-\lambda t} f(t)$ can be represented by the Fourier integral in the range $(0, \infty)$ provided that there exists a real number β such that $F(p)/p$ is a one-valued analytic function in the semi-plane

$$\operatorname{Re}(p) \leq \beta^*,$$

and

$$\frac{F(p)}{p} = O(p^{-n}), \quad (6)$$

when $|p|$ is great and n is positive and that λ is taken greater than or equal to β . Thus, the possibility of the representation of $e^{-\lambda t} f(t)$ in the Fourier series will be discriminated by virtue of the criterion upon $F(p)$ itself. It should be noted also that $F(p)/p$ can be represented in the Laplace integral,

$$\frac{F(p)}{p} = \int_0^\infty e^{-pt} f(t) dt, \quad \text{or} \quad F(p) \doteq f(t), \quad . . (7)$$

when $F(p)/p$ fulfils the above conditions.

Furthermore, we assume

$$f(0) = 0.$$

Then, if λ is taken sufficiently large, the graph of $e^{-\lambda t} f(t)$ will be such as to be shown in fig. 2, for example.

When this value of $e^{-\lambda t} f(t)$ in the range $(0, s)$ is considered as the first quarter of the above $g(t)$, $e^{-\lambda t} f(t)$ will be represented in such a sine series as (3).

From the curve of $e^{-\lambda t} f(t)$ in fig. 2, it seems permissible for practical purposes to neglect the contribution of the last integral in the following expression of b_n :—

$$b_n = \frac{2}{s} \int_0^s = \frac{2}{s} \int_0^\infty - \frac{2}{s} \int_s^\infty (8)$$

* $\operatorname{Re}(z)$ means the real part of z .

In order to interpret the analytic meaning of this neglect, it is necessary to call attention to some fundamental theorems in the operational calculus.

It will be seen immediately that when there exists the relation,

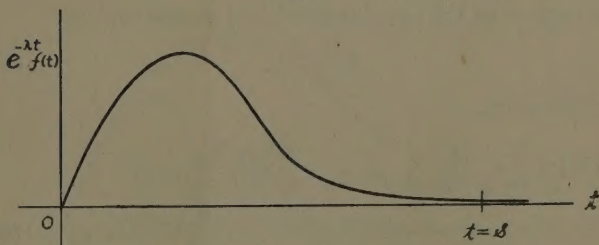
$$f(t) \doteq F(p),$$

for one value of p such that $R(p) \geq \beta$, then according as $R(k) > \beta$ or $R(k) < \beta$,

$$\int_0^t e^{-\lambda t} f(t) dt \doteq \frac{F(p+k)}{p+k} \quad . \quad . \quad . \quad (9)$$

for all p such that $R(p) > 0$, or $R(p) \geq \beta - R(k)$ respectively.

Fig. 2.



Especially when $\lambda > \beta$, we have

$$\int_0^t e^{-\lambda t} f(t) \sin vt dt \doteq \frac{1}{2i} \left\{ \frac{F(p+\lambda-i\nu)}{p+\lambda-i\nu} - \frac{F(p+\lambda+i\nu)}{p+\lambda+i\nu} \right\}, \quad . \quad . \quad . \quad (10)$$

provided that $R(p) > 0$ and that by virtue of Bromwich's relation

$$\int_0^s e^{-\lambda t} f(t) \sin vt dt = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{ps}}{p} \cdot \frac{1}{2i} \{ \} dp \quad . \quad (11)$$

where c is any positive constant and $\{ \}$ stands for that of (10).

When this complex integral is calculated by the so-called "method of steepest descent" or "Sattelpunkt-methode,"

$$\int_0^s = \frac{1}{2i} \left\{ \frac{F(\lambda-i\nu)}{\lambda-i\nu} - \frac{F(\lambda+i\nu)}{\lambda+i\nu} \right\} + \epsilon_\nu, \quad . \quad . \quad (12)$$

where

$$\epsilon_n = \frac{1}{2\pi i} \int_{\alpha-\lambda-i\infty}^{\alpha-\lambda+i\infty} \frac{e^{ps}}{p} \cdot \frac{1}{2i} \{ \} dp, \quad (\lambda > \alpha > \beta) \quad (13)$$

and the first term of the right side of (12) is the principal value of the integral, in this case the residue of the integrand at $p=0$.

Thus from (8)

$$b_n = \frac{2}{s} I \frac{F(\lambda - i\nu)}{\lambda - i\nu} + \frac{2}{s} \epsilon_n \dots \dots \dots (14)^*$$

However, the first term of the right side being nothing but the integral

$$\frac{2}{s} \int_0^\infty e^{-\lambda t} f(t) \sin \nu t dt,$$

the neglect of the last integral in (8) means that of

$$\beta_n = \frac{2}{s} \epsilon_n \dots \dots \dots (15)$$

Furthermore,

$$|\beta_n| < \frac{2}{\pi s} e^{-\mu s} \int_0^\infty \left\{ \frac{\mu^2 + y^2}{y^4 + 2(\mu^2 - \nu^2)y^2 + (\mu^2 + \nu^2)^2} \right\}^{\frac{1}{2}} \cdot \left| \frac{F(\alpha + iy)}{\alpha + iy} \right| dy, \quad (16)$$

where

$$\mu = \lambda - \alpha.$$

From this expression (16), it will be seen $|\beta_n|$ decreases like $e^{-\lambda s}$ when λs increases, and when it is intended to calculate the upper limit of $|\beta_n|$ from this, Simpson's rule will be used with profit for the quadrature of the various parts, as illustrated in the example (I).

Since in general problems λ can be taken so that we can neglect β_n , b_n can be written as follows:

$$b_n = \frac{2}{s} I \frac{F(\lambda - i\nu)}{\lambda - i\nu} \dots \dots \dots (17)$$

Thus Fourier constants will be calculated in the algebraic way from the given operational expression and $f(t)$ will be evaluated by means of the following expression:

$$f(t) = e^{\lambda t} \sum_n b_n \sin \frac{\pi n}{2s} t. \quad (n = \text{odd integer}). \quad (18)$$

* "I" stands for: "the imaginary part of."

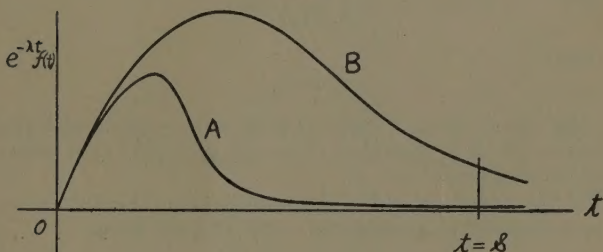
§ 3. On the Determination of λ .

The determination of the above-mentioned β is, as a rule, a comparatively easy matter, which is the same as in the Bromwich evaluation of the operational solution :

$$f(t) = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{pt} \frac{F(p)}{p} dp.$$

When β is determined, we may take any positive number greater than β as λ . In case β cannot be determined easily, we make a trial by giving λ comparatively large value. If this λ is too large the curve of $e^{-\lambda t} f(t)$ will be as A in fig. 3, and when λ is small it will be as shown by B. In the latter case, as the damping of $e^{-\lambda t} f(t)$ is

Fig. 3.



slow, the error of b_n due to the neglect of \int_s^∞ will be great.

The absolute value of this error will be estimated by means of (16). In the former case, that is as for A, many higher harmonics will rise in importance, and when we intend to calculate the function practically by Fourier series, many terms of the series should be taken. Thus, calculating b_n , if we find the decrease of the absolute values is not remarkable, it will be advisable to decrease λ a little more.

When we calculate a function by Fourier series in practice, the calculated result will deviate more or less from the true functional value, because we take only a finite number of terms of the series (in our case, $n=1, 3, \dots, 31$ in the following examples). This deviation of $e^{-\lambda t} f(t)$ and $\sum b_n \sin vt$ will take place in equal weight in the interval $0 \leq t \leq s$, but since we multiply $e^{\lambda t}$ in the

last calculation to obtain $f(t)$, the deviated value will be also multiplied by $e^{\lambda t}$, so it is inexpedient to take λ too large.

In our examples λs is taken about 4.5 to 8 ($e^{-\lambda s} = 0.01$ to 0.0003).

For the same reason we calculate ordinates of $f(t)$ up to $t=s/2$ instead of $t=s$. Of course this procedure does not limit the applicability of our method, since when we intend to know values of $f(t)$ in the range $0 \leq t \leq \tau$, we may take $s=2\tau$.

§ 4. *Illustrating Example I.*— $F(p)=1/p$.

Let us take $F(p)=1/p$ as the first illustrating example of the method mentioned above.

Since

$$\lim_{p \rightarrow \infty} F(p) = 0,$$

we have

$$f(0) = 0.$$

On the other hand, $F(p)/p = 1/p^2$ is a one-valued analytic function of all p , provided that the real part of p is any positive number.

Taking $\lambda s = 6$, $s = 30$ and consequently $\lambda = 0.2$, when b_n is calculated in accordance with (17), we have

$$b_n = \frac{e}{s} \text{I} \frac{F(\lambda - i\nu)}{\lambda + i\nu} = \frac{4\lambda\nu}{s(\lambda^2 + \nu^2)}, \quad \nu = \frac{n\pi}{2s},$$

and the following table will be obtained for each value of n :—

n	1.	3.	5.	7.	9.	11.
b_n	0.7650	1.000	0.5930	0.3220	0.1830	0.1110
n	13.	15.	17.	19.	21.	
b_n	0.0716	0.0487	0.0343	0.0250	0.0188	
n	23.	25.	27.	29.	31.	
b_n	0.0145	0.0114	0.0091	0.0074	0.0061	

When it is intended to calculate k ordinates of

$$y(t) = \sum b_n \sin vt$$

in the interval $(0, s/2)$, the m -th ordinate of y is written from (4) and

$$t_m = \frac{s}{2k} m \quad . \quad . \quad . \quad . \quad . \quad (19)$$

as follows :— $y_m = y(t_m) = \sum_n b_n \sin \frac{\pi m}{4k} n \quad . \quad . \quad . \quad . \quad . \quad (20)$

In our examples k is taken as 15, and the following table is calculated by means of b_n , tabulated above.

m	1.	2.	3.	4.	5.
y_m	0.7994	1.3596	1.6497	1.7868	1.8424
f_m	0.976	2.028	3.006	3.977	5.008
m	6.	7.	8.	9.	10.
y_m	1.8099	1.7237	1.6117	1.4959	1.3582
f_m	6.009	6.990	7.983	9.050	10.036
m	11.	12.	13.	14.	15.
y_m	1.2172	1.0890	0.9710	0.8553	0.7465
f_m	10.985	12.004	13.073	14.065	14.994

where $f_m = e^{\lambda t_m} y_m = e^{\lambda_m} y_m$.

Since $1/p \doteq f(t) = t$, each value of the above f_m , which is evaluated from the Fourier series in our method, shows good congruency with the corresponding true functional value $f(m) = m$.

Now let us illustrate the use of Simpson's rule in the calculation of the upper bound of $|\beta_n|$.

When we put $\alpha = \lambda - \alpha = 0.03$ and $n = 1$ in (16), since

$$\mu = \lambda - \alpha = 0.17, \quad \nu = 0.052,$$

we have

$$\begin{aligned} \phi &= \sqrt{\frac{y^2 + \mu^2}{y^4 + 2(\mu^2 - \nu^2)y^2 + (\mu^2 + \nu^2)^2}} \\ &= \sqrt{\frac{y^2 + 0.029}{y^4 + 0.052y^2 + 0.001}}, \end{aligned}$$

and
$$\left| \frac{F(\alpha + iy)}{\alpha + iy} \right| = \frac{1}{y^2 + \alpha^2} = \frac{1}{y^2 + 0.001}.$$

When the following first two integrals of the right side are calculated by Simpson's rule,

$$\int_0^\infty \frac{\phi}{y^2 + 0.001} dy = \int_0^1 + \int_1^{11} + \int_{11}^\infty,$$

we have

$$\int_0^1 = 251, \quad \int_1^{11} = 0.561,$$

and that since we may take

$$\phi = \frac{1}{y}, \quad \frac{1}{y^2 + 0.001} = \frac{1}{y^2},$$

in the range $y=11$ and $y=\infty$, so we have

$$\int_{11}^\infty = \int_{11}^\infty \frac{dy}{y^3} = 0.00415.$$

Thus

$$\int_0^\infty = 251.56.$$

Then from (16),

$$|\beta_1| < \frac{2e^{-(\lambda-\alpha)s}}{\pi s} \int_0^\infty = 0.0326.$$

On the other hand, when $|\beta_n|$ is calculated from true explicit function,

$$|\beta_n| = 2e^{-\lambda s} \left\{ \frac{\lambda}{\lambda^2 + \nu^2} + \frac{\lambda^2 - \nu^2}{s(\lambda^2 + \nu^2)^2} \right\},$$

and

$$|\beta_1| = 0.0273 < 0.0326.$$

§ 5. The Evaluation of $f(t)$ in case $f(0) \neq 0$.

It is assumed, hitherto, that $f(0)=0$. Let us now remove this restriction, and let $f(t)$ be a function such that when we take an appropriate positive integer m , a function, $h(t)$, defined by the following expression,

$$h(t) = t^m f(t), \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

becomes zero at $t=0$.

Generally speaking, when we intend to represent a function which is not zero at $t=0$ in the sine series, we

shall be obliged to take many more terms of the series than when the function is zero at $t=0$. In such cases, therefore, it will be wise to calculate first by the sine series an auxiliary function which is zero at $t=0$, and then to find the value of the given function from its known relation with the auxiliary function.

Now let us take $h(t)$ of (21) as this auxiliary function of $f(t)$. Then by virtue of the relation given by van der Pol*, there is the following relation between $F(p)$ and the operational expression $H(p)$ of $h(t)$:

$$H(p) = (-1)^m p \frac{d^m}{dp^m} \left\{ \frac{F(p)}{p} \right\} (22)$$

Applying the method of evaluation mentioned in preceding articles to this $H(p)$, we can obtain $h(t)$ in the following form :

$$h(t) = e^{\lambda t} \sum b_n \sin vt.$$

Thus by virtue of (21), $f(t)$ will be evaluated as follows :

$$f(t) = \frac{e^{\lambda t}}{t^m} \sum b_n \sin vt. (23)$$

The procedure of this article is useful not only in case $f(t) \neq 0$, but also in case b_n has slow convergency notwithstanding $f(t) = 0$.

§ 6. Illustrating Examples II.

(i.) $f(o) = \text{finite number}$.

Let us take

$$F(p) = \frac{p}{\sqrt{p^2 + 1}},$$

then

$$\lim_{p \rightarrow \infty} \frac{p}{\sqrt{p^2 + 1}} = f(o) = 1.$$

Taking $h(t) = tf(t)$, we have by (22),

$$H(p) = -p \cdot \frac{d}{dp} \left\{ \frac{1}{\sqrt{p^2 + 1}} \right\} = \frac{p^2}{(p^2 + 1)^{3/2}}, . . . (24)$$

and

$$\lim_{p \rightarrow \infty} H(p) = h(o) = 0.$$

Putting

$$\begin{aligned} \lambda s - i\nu s &= \rho_n e^{-i\theta_n}, \\ (\lambda s - i\nu s)^2 + s^2 &= \zeta_n e^{-i\varphi_n}, \end{aligned}$$

* Balth. van der Pol, Phil. Mag. viii. p. 863 (1929).

by virtue of (17), b_n will be obtained as follows :

$$b_n = 2sI \frac{\lambda s - i\nu s}{\{(\lambda s - i\nu s)^2 + s^2\}^{3/2}} \\ = \frac{2s\rho_n}{\zeta^{3/2}} \sin(\frac{3}{2}\phi_n - \theta_n).$$

Hence we obtain the following tables, taking $s=15$, $\lambda s=4.5$:

n	1.	3.	5.	7.
b_n	-0.0093	-0.0287	-0.0471	-0.0308

n	9.	11.	13.	15.	17.	19.
b_n	0.1552	0.2540	0.1333	0.0646	0.0349	0.0209

n	21.	23.	25.	27.	29.	31.
b_n	0.0136	0.0094	0.0069	0.0050	0.0039	0.0031

and

m	1.	2.	3.	4.	5.
y_m	0.3943	0.5743	0.4899	0.2447	-0.0564
h_m	0.458	0.775	0.768	0.446	-0.120
$f\left(\frac{m}{2}\right)$	0.916	0.775	0.512	0.223	-0.048
$J_0\left(\frac{m}{2}\right)$	0.938	0.765	0.512	0.224	-0.048

m	6.	7.	8.	9.	10.
y_m	-0.3128	-0.4661	-0.4797	-0.3699	-0.1948
h_m	-0.769	-1.332	-1.593	-1.427	-0.873
$f\left(\frac{m}{2}\right)$	-0.256	-0.380	-0.398	-0.317	-0.175
$J_0\left(\frac{m}{2}\right)$	-0.260	-0.380	-0.397	-0.321	-0.178

m	11.	12.	13.	14.	15.
y_m	-0.0085	0.1475	0.2399	0.2535	0.2043
h_m	-0.044	0.892	1.686	2.070	1.938
$f\left(\frac{m}{2}\right)$	-0.008	0.149	0.260	0.296	0.258
$J_0\left(\frac{m}{2}\right)$	-0.007	0.151	0.260	0.300	0.266

where

$$y_m = \sum b_n \sin \frac{\pi}{60} mn,$$

$$t_m = \frac{s}{2k} \cdot m = \frac{m}{2},$$

$$h_m = h(t_m) = e^{\lambda t_m} y_m = e^{0.15m} y_m,$$

$$f\left(\frac{m}{2}\right) = \frac{2}{m} \cdot h\left(\frac{m}{2}\right),$$

$J_0(t)$ = the Bessel coefficient of order zero.

The above results show the good congruency of $f(t)$ evaluated by our method with the true explicit function $J_0(t)$ of $F(p)$.

(ii.) $f(0) = \infty$.

Let us take

$$F(p) = \sqrt{p},$$

then

$$\lim_{p \rightarrow \infty} F(p) = f(0) = \infty.$$

However, since the order of infinity is $\frac{1}{2}$, when we make $m > \frac{1}{2}$ in (21), it will be that

$$h(0) = 0.$$

Especially, if $m=1$,

$$H_1(p) = -p \frac{d}{dp} \frac{\sqrt{p}}{p} = \frac{1}{2} \frac{1}{\sqrt{p}},$$

and if $m=2$,

$$H_2(p) = (-1)^2 p \frac{d^2}{dp^2} \frac{\sqrt{p}}{p} = \frac{3}{4} \frac{1}{p\sqrt{p}}.$$

When we intend to calculate $I\{H(\lambda - i\nu)/(\lambda - i\nu)\}$ with respect to $H_1(p)$ or $H_2(p)$, it seems that b_n of $H_2(p)$ will decrease more rapidly than that of $H_1(p)$.

Hence taking

$$h(t) \doteq H(p) = \frac{1}{p\sqrt{p}},$$

we calculate

$$\sqrt{p} \doteq f(t) = \frac{3h(t)}{4t^2}.$$

In this calculation s and λs are taken 30 and 7.5 respectively, and the following results are obtained :—

n	1.	3.	5.	7.	9.	11.
b_n	0.9981	1.3877	0.7613	0.3317	0.1347	0.0504
n	13.	15.	17.	19.	21.	
b_n	0.0141	-0.0016	-0.0082	-0.0106	-0.0111	
n	23.	25.	27.	29.	31.	
b_n	-0.0107	-0.0100	-0.0092	-0.0084	-0.0076	

and

m	1.	2.	3.	4.	5.
y_m	0.6111	1.2617	1.8461	2.2307	2.4053
h_m	0.785	2.080	3.908	6.064	8.395
$f(m)$	0.588	0.390	0.326	0.284	0.252
$\frac{1}{\sqrt{\pi m}}$	0.564	0.399	0.326	0.282	0.252
m	6.	7.	8.	9.	10.
y_m	2.4569	2.4269	2.3112	2.1365	1.9502
h_m	11.011	13.966	17.078	20.271	23.758
$f(m)$	0.230	0.214	0.200	0.188	0.178
$\frac{1}{\sqrt{\pi m}}$	0.230	0.213	0.200	0.188	0.178
m	11.	12.	13.	14.	15.
y_m	1.7588	1.5614	1.3635	1.1914	1.0358
h_m	27.512	31.362	35.165	39.454	44.043
$f(m)$	0.170	0.163	0.156	0.151	0.147
$\frac{1}{\sqrt{\pi m}}$	0.170	0.163	0.156	0.151	0.146

where

$$y_m = \sum b_n \sin vt_m,$$

$$t_m = \frac{s}{2k} m = m,$$

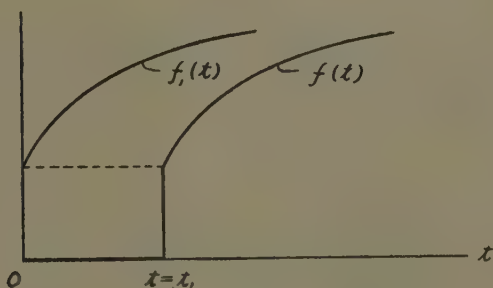
$$h_m = h(t_m) = e^{0.25m} y_m, \quad f(m) = \frac{3}{4} \frac{h(m)}{m^2},$$

$$\frac{1}{\sqrt{\pi t}} \doteq F(p) = \sqrt{p}.$$

Thus our method of evaluation gives the very successful result showing good congruency with the explicit function, $1/\sqrt{\pi t}$ of the operational expression \sqrt{p} .

Moreover, when $f(t)$ is zero everywhere in the range $t=0$ and $t=t_1$, as in fig. 4, it will be wise to calculate first

Fig. 4.



$f_1(t)$, which is the mere shifted function of $f(t)$ towards the left, from the following operational expression $F_1(p)$ of $f_1(t)$,

$$f_1(t) \doteq F_1(p) = e^{t_1 p} F(p),$$

by means of the method mentioned above.

§ 7. Summary.

In the preceding articles we have shown the theoretical ground and the practical calculation of our Fourier series evaluation which consists in the calculation of Fourier coefficients from the given Heaviside operational expression in order to evaluate its explicit function in the range from $t=0$ to some other arbitrary but fixed value of t .

Since Fourier coefficients can be obtained here from the given operational expression by the mere algebraic calculation of complex numbers, this method will always be practicable, even if the other methods fail in the evaluation of the explicit function.

In the present paper only simple examples are adopted to illustrate our method of evaluation in cases where $f(o)=0$, $f(o)=a$ finite number, and $f(o)=\infty$. It is intended that physical and technical applications of this method will be shown in future.

In conclusion, the author wishes to offer his sincere thanks to Dr. K. Kurokawa, Professor of the Waseda University, to whom the author is deeply indebted for many suggestions and helpful criticism throughout the progress of this research.

XC. *On the Measurement of the Thickness of Thin Transparent Crystalline Plates.* By K. S. SUNDARARAJAN, Research Scholar, Indian Association for the Cultivation of Science, Calcutta *.

[Plate XIX.]

1. Introduction.

IN the September number of the 'Proceedings of the Physical Society' †, Darbyshire has described an elegant method for determining accurately the thickness of thin transparent plates. Using a spectroscope and incident white light, Darbyshire photographed in the usual manner the Edser-Butler fringes by placing the plate in front of the spectroscope slit, and the Talbot bands by placing the plate between the prism and the collimator of the spectroscope. As is well known, the positions of the maxima of intensity of the Edser-Butler fringe system are given by the condition, $2\mu d$ is an integral multiple of λ , where μ is the refractive index of the plate for the wave-length λ and d is its thickness. On the other hand, the positions of the maxima of the Talbot band system correspond to the condition, $(\mu-1)d$ is an integral multiple of λ . If, therefore, m is the number of Edser-Butler fringes between any two known wave-

* Communicated by Prof. K. S. Krishnan, D.Sc.

† Vol. xlv. p. 626 (1934).

lengths λ_1 and λ_2 , and n is similarly the number of Talbot bands between the same wave-lengths, it is easily seen that

$$d = \frac{m - 2n}{2(\lambda_2^{-1} - \lambda_1^{-1})} \quad (1)$$

Since m and n are obtained from the two spectrograms, d can be readily calculated.

In connexion with some measurements on the refractive indices of organic crystals, which are usually available in the form of small thin flakes only, we have been using in this laboratory a similar optical method for the measurement of the thickness of these plates. The experimental arrangement adopted by us is specially suited for measurements with plates of small area, 1 mm. \times $\frac{1}{2}$ mm. or less. The following account of the method may be of interest.

2. Theory of the Method.

While attempting to photograph the Talbot bands with the optical arrangement to be described in the next section, it was observed that, superposed on the Talbot bands, there appeared another system of interference bands having the same spacing as the Edser-Butler fringe system. The occurrence of these bands is easily explained. According to the usual theory of the Talbot bands, one half of the wave, due to its passage through the plate, is retarded relatively to the other half. The interference between these two halves gives rise to the maxima and minima of the Talbot system. In this explanation it is tacitly assumed that the transparent plate merely introduces a phase retardation, without affecting the *amplitude*; whereas actually, owing to multiple internal reflexions in the plate, the amplitude of the wave transmitted by the plate will vary with the wave-length; the amplitude is a maximum or a minimum according as $2\mu d$ is an even or odd multiple of $\lambda/2$. Due to this fluctuation in the amplitude of one of the interfering beams, there will appear, superposed on the Talbot bands, another set of fringes, whose maxima and minima will be defined by the above conditions. The positions of these latter fringes are evidently identical with those of the usual Edser-Butler fringes of the transparent plate.

The simultaneous occurrence of both the band systems considerably simplifies the experiment.

3. *Experimental Arrangement.*

For obtaining the Talbot bands with the usual arrangement, plates of a fairly large area have to be used. But in the case of organic crystals, plates of such large area of uniform thickness are very rare, the majority of them being about $1\text{ mm.} \times \frac{1}{2}\text{ mm.}$ or less. The following arrangement was found to be very suitable for such crystals. A horizontal slit backed by a filament lamp formed the source of white light. The light issuing from the slit was collimated by an achromatic long focal length lens, and the parallel beam was passed through a square aperture the upper or lower half of which was covered by the crystal flake, with one of its natural edges horizontal. The parallel beam of light issuing out of the aperture passed through a nicol suitably oriented so as to have its principal plane parallel to one of the extinction directions of the crystal plate, and was then focussed by a second achromatic lens on the slit of a spectrograph. This arrangement evidently corresponds to an echelon of two steps which is *crossed* with the spectrograph, and the Talbot bands obtained through the spectrograph will naturally be inclined *. Superposed on the Talbot bands there appeared a number of vertical fringes of the Edser-Butler type, as contemplated by the theory detailed in the previous section. The area of the aperture covered by the crystal plate is suitably adjusted so as to give the best visibility for the Talbot bands. By photographing the above spectrum, along with a comparison spectrum of the copper arc, the number of Talbot bands n , and of the Edser-Butler fringes m , appearing between any two selected wave-lengths, can be easily determined, and hence d can be calculated with the help of relation (1).

In the usual arrangement for obtaining the Talbot bands, where the "two step echelon," instead of being crossed with the spectrograph as in the above arrangement, is parallel to it, there is a certain optimum thickness of the plate for which the Talbot bands have the best visibility; whereas the present arrangement has the advantage pointed out by Sethi, namely, the visibility of the inclined Talbot bands is independent of the thickness of the plate.

* Inclined Talbot bands of the above type have been studied in detail by N. K. Sethi (Phys. Rev. xvi. p. 519 (1920)), and more recently by A. C. G. Beach (Proc. Phys. Soc. xlv. p. 474 (1933)).

In fig. 1 (Pl. XIX.) is reproduced a spectrogram obtained with a thin plate of the monoclinic crystal chrysene by the above method; the plate was parallel to the (001) plane. The nicol was oriented so as to transmit light vibrations parallel to the "b" axis, which naturally is one of the extinction directions of the crystal plate. The Edser-Butler type of fringes superposed on the Talbot bands are quite clearly seen in the spectrogram.

When the second lens in the above optical arrangement, which focusses the light on the spectroscope slit, is removed, the fringes that are observed through the spectroscope are very similar to those observed before, with this difference, however, that the Edser-Butler type fringes appear only in the upper or lower half of the spectrum depending on which half of the aperture is covered by the crystal plate. Fig. 2 (Pl. XIX.) gives a spectrogram taken under these conditions of the same chrysene plate (which covered the lower half of the aperture, whereas it covered the upper half in the arrangement corresponding to fig. 1). The thickness of the crystal plate as determined from these spectrograms is found to be $5.18(\pm 0.01) \times 10^{-3}$ cm.

The author expresses his thanks to Prof. K. S. Krishnan, E.Sc., for his kind interest in the work.

XCI. *Investigations of Raman Spectra.—Raman Spectra of Dekahydro and Tetrahydro-naphthalene.* By S. K. MUKERJI *.

[Plate XX.]

Abstract.

THE Raman spectra of dekahydro and tetrahydro naphthalene have been studied carefully with a view to recording the faint lines and assigning the frequencies correctly. Solutions of quinine sulphate and *m*-dinitro benzene in benzene have been used separately as light filters. The latter solution, as was reported previously by R. Bär †, has proved to be very efficient in absorbing

* Communicated by the Author.

† *Helv. Phys. Acta*, v. p. 174 (1932).

the light both of the 4046 group of lines of the mercury arc, and also of the shorter wave-lengths and has helped considerably in the correct assignment of the frequencies.

The examination of dekalin has yielded seventeen new lines not recorded before at wave-numbers 19,707, 19,761, 20,046, 21,588, 21,665, 21,881, 22,005, 22,495, 22,562, 22,586, 22,614, 22,614, 22,765, 22,784, 23,082, 23,098, and 23,173.

In tetralin twelve new lines at wave-numbers 19,892, 19,998, 21,833, 21,864, 22,239, 22,715, 23,100, 23,203, 23,814, 24,001, 24,371, and 24,446, have been found.

In both dekalin and tetralin three anti-Stokes lines not recorded before at frequencies 596, 494, 407, and 162, 265, 1433 cm^{-1} respectively, have been found.

Three frequencies 2922, 2892, and 2855 cm^{-1} due to dekahydro naphthalene, only two of which were previously reported by G. B. Bonino and P. Cella*, and which are characteristic of the carbon hydrogen linking in the $:\text{CH}_2$ groups, and one at 1447 characteristic of the aliphatic groups also reported by Bonino and Cella, and which are all present in the cyclo hexane spectrum, have been obtained.

Two other lines at frequencies 443 and 376 cm^{-1} not reported before, and which also appear in the cyclo hexane spectra, have been obtained. Six other lines previously reported by Bonino and Cella which appear in the cyclo-hexane and naphthalene spectra have also been obtained.

Tetralin has given a large number of Raman lines including the one at 3046 cm^{-1} , characteristic of the aromatic carbon hydrogen linking and another at 1582 cm^{-1} , characteristic of the aromatic linking $\text{C}=\text{C}$. Five other lines, which are present in the naphthalene spectrum and are due to the CH groups, have been obtained, and they all confirm Bonino and Cella's results. Two new lines not recorded before at frequencies 2940 and 699 cm^{-1} , which are also present in the cyclo hexane spectrum, have been observed.

The author's results are compared with those of Bonino and Cella and with the cyclo hexane and naphthalene spectra, and discussed.

* *Rendiconti Reale Accad. Roma*, xiii. p. 784 (1931); and *Lincae Atti*, xv. pp. 572-576 (April 3, 1932).

1. *Introduction.*

WHENEVER a substance has a large number of Raman lines, and if no suitable filter be used to cut off the light of the 4046 group of lines of the mercury arc, and if only one strong mercury line be used as the exciting line, *e. g.*, the 4358.8 Å. mercury line, there is always the possibility of wrong assignment. Hence, in the case of such substances it is most desirable to use a suitable filter to cut off the 4046 group of mercury lines. In the present investigation of the Raman effect of dekalin and tetralin, which has so far been examined only by Bonino and P. Cella, both these liquids have given a large number of Raman lines. Several filters were tried in order to cut off or weaken considerably the 4046 Å. region of the spectrum, and ultimately a filter of *m*-dinitro benzene dissolved in suitable proportions in benzene was found to be very efficient in weakening considerably the 4046 region of the mercury arc. This filter had previously been very successfully used by R. Bär *, using carbon tetrachloride as the liquid for scattering the light. In the case of tetralin two spectra were obtained, one with a quinine sulphate filter, which only partially weakened the 4046 region, and the other with a filter of *m*-dinitro benzene dissolved in benzene. This latter filter weakened considerably the light of the 4046 region and more of the shorter wave-length radiations. A comparison of the two plates, and also a consideration of the relative intensities of the lines on each plate, gave the origin of each Raman line. In the case of dekalin only one plate was obtained using the latter filter. Both these liquids have given practically all the lines reported by Bonino and Cella and, in addition, a large number of new lines, including three anti-Stokes lines not recorded before. The results are discussed in the present paper.

2. *Experimental.*

The liquids dekalin and tetralin, obtained from the chemical laboratory of the Agra College, India, were subjected to repeated distillation and the middle portion of the distillate was used, especial care being taken to make the liquids dust-free. The liquid was contained

* *Loc. cit.*

in a glass tube, 8 inches long and $1\frac{1}{2}$ inches in diameter, drawn out at the open end into a horn which was painted black on the outside so as to furnish a dark background. The other parts of the tube were also painted black, leaving an aperture 8 mm. square at the other end of the tube, which was convex and served as an observation window, and another aperture, 6" by 1", along the length of the tube at the top for illumination. A six-inch quartz mercury-vapour lamp of the flat type was used

TABLE I.
Dekalin frequencies
(with *m*-dinitro benzene in benzene filter).

Number.	$\Delta\nu$ in cm. ⁻¹ .	Intensity I.	Number.	$\Delta\nu$ in cm. ⁻¹ .	Intensity I.
1....	3236	0	20....	851	4
2....	3177	0	21....	803	3
3....	2922	5	22....	763	5
4....	2892	1	23....	596 (broad)	1
5....	2855	4	24....	494	3
6....	2658	0	35....	443	1
7....	1447	5	26....	407 (broad)	3
8....	1362	1	27....	376	1
9....	1350	1	21....	324	0
10....	1273	1	30....	294	0
11....	1256	3	30....	294	0
12....	1166	3			
13....	{ 2859	5	31....	173	0
	{ 1092	3	32....	156	0
14....	1057	2	33....	1623	00
15....	1048	3	34....	1607	00
16....	1024	1	35....	1448	2
17....	991	2	36....	407	2
18....	993	2	37....	494	1
19....	780	1	38....	596	0

as the source of light, and the liquid filter contained in a flat and wide glass bottle was placed between the tube and the lamp, the three being as close as possible to one another. The glass bottle was, in fact, resting on the tube containing the liquid. Special care was taken to shield all extraneous light by suitable screens and apertures, and the genuinely scattered light was focussed on to the slit of the spectrograph, since any stray light only tends to an over-exposure and the consequent broadening of the Rayleigh lines. The spectrograms were obtained with a fairly rapid Fuess glass spectrograph, having a dispersion of about 16 Å. per mm.

in the region of 4358 Å. The slit of the spectrograph was kept very narrow in order to obtain the maximum possible resolution. Ilford Golden Isozenith plates, speed D. 700, were used for photographing the scattered spectra. These plates were backed before use to eliminate the halation accompanying the strong lines. The times of exposure varied from 48 hours to 72 hours, and the lamp was continuously cooled by a fan.

TABLE II.

Tetralin frequencies (with quinine sulphate filter).

Number.	$\Delta\nu$ in cm. ⁻¹ .	Intensity I.	Number.	$\Delta\nu$ in cm. ⁻¹ .	Intensity I.
1....	3046	5	26....	582	3
2....	2940	4	27....	511	3
3....	2918	4	28....	430	4
4....	2865	4	29....	265	2
5....	1602	3	30....	223	0
6....	1582	3	31....	162	3
7....	1458	3	32....	162	$\frac{1}{2}$
8....	1433	5	33....	265	0
9....	1376	6	34....	1448	1
10....	3114	2	35....	1376	4
11....	3057	2	36....	1341	2
12....	1290	1	37....	1202	4
13....	1283	1	38....	1174	0
14....	1205	5	39....	1037	7
15....	2948	5	40....	891	2
16....	{ 2928	5	41....	826	0
	{ 1161	2	42....	762	1
17....	1105	0	43....	727	5
18....	1074	0	44....	704	1
19....	1037	6	45....	579	3
20....	1038	1	46....	561	2
21....	839	0	47....	507	3
22....	814	2	48....	435	3
23....	752	2	49....	1433	5
24....	723	5	50....	259	2
25....	699	2			

The plates were measured on a Hilger cross-slide photo-measuring micrometer reading to a thousandth of a millimetre. For comparison and checking the results the spectrum of an iron arc was used. The wavelengths were calculated using the well-known Hartmann's dispersion formula.

The results are given in Tables I., II., and III. Complete tables showing assignments are given at the end of the paper and the microphotometer records in figs. 2 and 3.

TABLE III.

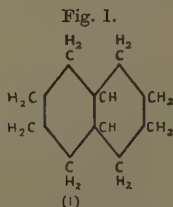
Tetralin frequencies
(with *m*-dinitro-benzene in benzene filter).

Number.	$\Delta\nu$ in cm. ⁻¹ .	Intensity I.	Number.	$\Delta\nu$ in cm. ⁻¹ .	Intensity I.
1....	3046	4	18....	839	0
2....	2940	3	19....	814	2
3....	2918	2	20....	752	2
4....	2865	2	21....	723	4
5....	1602	2	22....	699	0
6....	1582	2	23....	582	2
7....	1458	2	24....	511	2
8....	1433	3	25....	430	3
9....	1376	4	26....	265	12
10....	{ 1290	1	27....	223	0
	{ 3057	1	28....	162	3
11....	1283	1	29....	162	2
12....	1205	4	30....	265	0
13....	1161	1	31....	1376	2
14....	1105	0	32....	1037	2
15....	1074	0	33....	727	2
16....	1037	4	34....	1433	00
17....	1038	0			

3. Discussion of Results.

Dekalin.

Dekalin was studied previously by G. B. Bonino and P. Cella*. The frequencies recorded by them are given in Table IV., together with those obtained by the

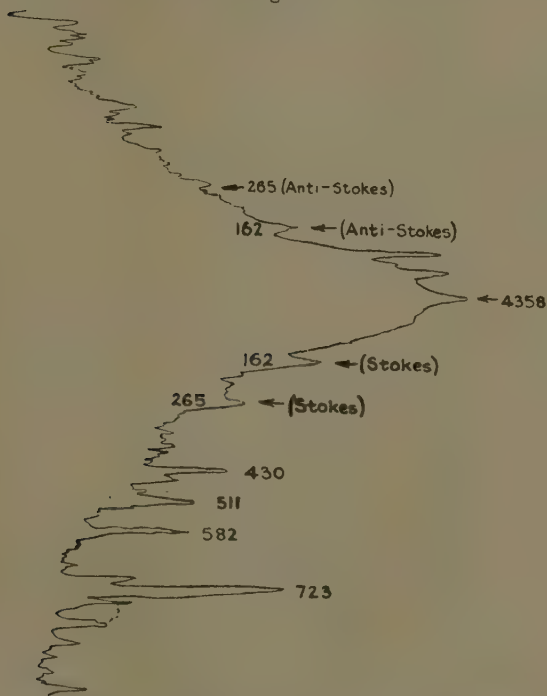


author for a comparative study. It will be seen from this table that seventeen new lines have been obtained by the author which were not recorded before. Bonino and his collaborator have reported altogether 35 Raman lines and, according to them, the last sixteen of these

* *Loc. cit.*

are due to the 4046 Å. line of the mercury arc. Only one spectrogram was taken by the author with *m*-dinitrobenzene in benzene filter, and hence these last sixteen lines reported by these authors could be expected to be present very weakly, if at all, on the author's plate. Of the first nineteen lines sixteen have been reported by these

Fig. 2.

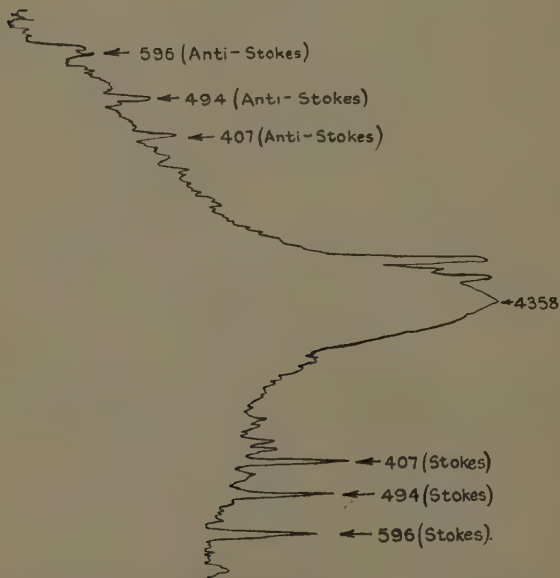


Microphotometer records of the Raman spectra of tetralin.

authors as due to 4358 Å. excitation. It is significant to remark here that all these sixteen lines have been obtained by the author on the *m*-dinitrobenzene (in benzene filter) plate (Pl. XX. (c)). One more line at wave-number 21,846, which is a strong line on Bonino and Cella's plate, and which they have written as due to 4046 Å. excitation having a frequency shift of 2859 cm.^{-1} , has also appeared in the

present investigation and is fairly strong. It thus appears that this line is due to both 4046 Å. and 4358 Å. excitations. The other two, one at wave-number 21,628 and frequency shift 1367 cm^{-1} , which is very weak, and the other at 21,780 and frequency shift 2925 cm^{-1} , which is strong (according to Bonino and Cella), were not obtained by the author. This confirms those authors' conclusion that the latter frequency is due

Fig. 3.



Microphotometer records of the Raman spectra of dekaline.

to 4046 Å. excitation. The line at wave-number 21,881 has appeared comparatively weakly on the present author's plates, although it is quite strong on those of Bonino and Cella. This indicates that this line has also been excited both by 4046 Å. and 4358 Å.

Of the last sixteen lines four have also appeared on the author's plate, and they lie in the region between the 4046 Å. and 4358 Å. lines of the mercury arc. Hence

they could either be due to 4046 Å. excitation or they might be the anti-Stokes lines due to 4358 Å. One of these at wave-number 23,257 and frequency shift 1448 cm.⁻¹ is a strong line according to Bonino and Cella,

TABLE IV.
Dekalin frequencies
(with *m*-dinitro benzene in benzene filter).

Number.	G. B. Bonino and P. Cella.	S. K. Mukerji.	Number.	G. B. Bonino and P. Cella.	S. K. Mukerji.
	—	3236	18....	493	494
	—	3177		—	443
1....	2923	2922	19....	406	407
	—	2892	—	—	376
2....	2854	2855	—	—	352
	—	2658	—	—	324
3....	1452	1447	—	—	294
4....	1363	1362	—	—	173
5....	1367	—	—	—	156
	—	1350	—	—	1623
	—	1273	—	—	1607
6....	1258	1256	20....	1445	1448
7....	1168	1166	21....	1358	407
8....	2925	—	22....	1262	494
9....	2859	{ 2859	23....	1161	596
	—	{ 1092	24....	1044	—
	—	{ 1057	25....	1006	—
10....	1048	1048	26....	985	—
11....	1017	1024	27....	933	—
12....	990	991	28....	872	—
	—	933	29....	850	—
13....	893	880	30....	800	—
14....	851	851	31....	753	—
15....	808	803	32....	592	—
16....	753	763	33....	554	—
17....	594	596	34....	486	—
			35....	398	—

and fairly strong on Pl. XX. (c). This line does not show any corresponding strong Stokes line. Hence it is due to 4046 Å. excitation as reported by Bonino and Cella. The other three lines at wave-numbers 23,534, 23,432, 23,345, and frequency shifts 596, 494, and 407 cm.⁻¹ respectively are comparatively weak, but are of appreciable intensity as shown on Pl. XX. (c). They are the anti-Stokes lines due to 4358 Å., and the Stokes lines corresponding to them and of sufficient intensity are found at the

corresponding wave-numbers 22,342, 22,444, and 22,531 respectively.

Two other very weak new lines not recorded before also fall within the region 4046 Å. and 4358 Å. of the mercury arc. They do not show any corresponding Stokes lines and they are due to 4046 Å excitation.

The other new frequencies obtained which have not been recorded before are at 3236, 3177, 2892, 2658, 1350 1273, 1057, 933, 443, 376, 352, 324, 294, 173, and 156 cm^{-1} respectively. The remaining frequencies obtained, as will be seen from Table IV., are only those which were previously obtained by Bonino and Cella, and hence they only confirm these authors' results.

The three frequencies 2922, 2892, and 2855 cm^{-1} obtained by the author, two of which were previously obtained by Bonino and Cella and which are characteristic of the carbon hydrogen linking in the :CH₂ group, have been found to coincide remarkably well with those due to cyclo-hexane obtained by P. Krishnamurti *. One new weak line at 2658 cm^{-1} has also been obtained. It is found to be excited by 4358 Å., and it falls in a region of the spectrum where there is no confusion due to the presence of a large number of lines. Its wave-length is quite different from those of the mercury bands in this region, and also agrees closely with 2662 cm^{-1} calculated by P. Krishnamurti † for cyclo-hexane. Another line at 1447 cm^{-1} , due to deformation oscillations of the :CH₂ group, has been reported by Bonino and Cella. It is characteristic of the aliphatic groups, being present in all aliphatic compounds containing hydrogen. It is also present in the cyclo-hexane spectrum, and its presence here confirms the result of Bonino and Cella. Moreover, six other frequencies, also reported by Bonino and Cella, at 1362, 1256, 1166, 1024, 991, and 596 cm^{-1} which are common to both cyclo-hexane and naphthalene spectra have been obtained. The line obtained at 851 cm^{-1} is found to coincide very exactly with the fairly strong Raman frequency 851 due to benzene. This has evidently to be attributed to the oscillations of the carbon atoms. Another line at 803 cm^{-1} which, according to Ganesan and Venkateswaren ‡ and

* P. Krishnamurti, Ind. Journ. Phys. vi. p. 543 (1932).

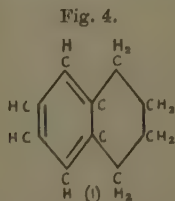
† *Loc. cit.*

‡ Ganesan and Ventateswaren, Ind. Journ. Phys. iv. p. 195 (1929).

also P. Krishnamurti, is very prominent and sharp in the cyclo-hexane spectrum, has also been obtained and is fairly strong. This line has also been reported by Bonino and Cella. Two other new lines have also been obtained by the author at frequencies 443 and 376 cm^{-1} respectively, and they correspond to 425 and 381 cm^{-1} , due to cyclo-hexane obtained by P. Krishnamurti. It may be noted here that dekaline has given one line at 991 cm^{-1} which is characteristic of benzene and is due to the hexagonal carbon ring. This frequency appears in the Raman spectra of almost all the benzene derivatives. Bonino and Cella have also reported this line, but its existence is not explained. It should, perhaps, be associated also with the oscillations of the carbon atoms. All these results are tabulated in Table VI., along with those due to Bonino and Cella. The cyclo-hexane frequencies due to Ganesan and Venkateswaren, and to P. Krishnamurti, and the naphthalene frequencies due to R. Bär * have been added in this table for a comparative study.

4. *Tetralin.*

Tetralin was also studied previously by Bonino and Cella †. Their results are given in Table V., together with those of the author for comparison. It will be



seen from this table that many lines not recorded by Bonino and his collaborator have been observed by the author. Altogether 41 lines have been recorded by Bonino and Cella. Of these 38 have been obtained by the author, and they agree remarkably well with those of Bonino and Cella. The lines at wave-number shifts 2857 (2), 938 (0), and 1284 (2) recorded by these authors were not obtained during the present work. The new

* R. Bär, 'Nature,' exxiv. p. 692 (Nov. 2, 1929).

† *Loc. cit.*

frequencies obtained by the author are at 3046, 2940, 1105, 1074, 699, 223, 162, 265, 891, 704, 1433, and 259 cm^{-1} respectively. Two weak lines at wave-numbers 21,833 and 21,864 were observed on both of the writer's plates (one obtained with a quinine sulphate filter and the other obtained with *m*-dinitro-benzene in benzene filter). These lines, therefore, appear to be due to 4358 Å. and

TABLE V.

Tetralin frequencies (with quinine-sulphate filter).

Number.	G. B. Bonino and P. Cella.	Mukerji.	Number.	G. B. Bonino and P. Cella.	Mukerji.
—	—	3046	23....	583	582
—	—	2940	24....	512	511
1....	2020	2918	25....	430	430
2....	2867	2865	26....	265	265
3....	1602	1602	—	—	223
4....	1583	1582	27....	158	162
5....	1458	1458	—	—	162
6....	1433	1433	—	—	265
7....	1373	1376	28....	1454	1448
8....	1338	3114	29....	1376	1376
9....	3049	3057	30....	1342	1341
10....	1282	1290	31....	1284	—
11....	1283	1283	32....	1205	1202
12....	1205	1205	33....	1161	1174
13....	2950	2948	34....	1037	1037
14....	2925	2928	—	—	891
15....	2857	—	35....	817	826
—	—	1105	36....	762	762
—	—	1074	37....	726	727
16....	1038	1037	—	—	704
17....	1035	1038	38....	587	579
18....	938	—	39....	567	561
19....	848	839	40....	515	507
20....	814	814	41....	435	435
21....	756	752	—	—	1433
22....	723	723	—	—	259
—	—	699			

not 4046 Å., as there was hardly any difference in the intensity of these lines on the two plates (Pl. XX. (a) and (b)). In between these two lines is the line at wave-number 21,848 recorded by Bonino and Cella as due to 4046 Å. excitation which has not been obtained by the author.

Tetralin has given three anti-Stokes lines not recorded before at wave-numbers 23,100, 23,203 and 24,371. and frequency shifts 162, 265, and 1433 cm^{-1} respectively, due to 4358 Å. mercury line, the corresponding Stokes lines being at wave-numbers 22,776, 22,673,

and 21,505 respectively. They fall within the region 4046 Å. and 4358 Å. of the mercury arc, and the first two are distinctly visible on both of the plates. The last one, however, is extremely feeble on the *m*-dinitro benzene in benzene filter plate, although it is quite distinct on the quinine sulphate filter plate. They are therefore due to 4358 Å. excitation. The Stokes lines are found exactly at the corresponding wave-numbers mentioned above. The anti-Stokes line at frequency shift 1433 cm.^{-1} , however, being extremely weak on the author's *m*-dinitrobenzene in benzene filter plate requires further confirmation. The dekaline and tetralin spectrograms are reproduced in Pl. XX., and the positions of the Stokes and anti-Stokes lines are shown on the micrometer records, and will be found at the end of the paper.

The presence of the aromatic ring and the carbon hydrogen linking in the molecule in addition to the aromatic linking $\text{C}=\text{C}$ are necessarily the cause of a large number of Raman lines in this substance, indicating that all sorts of oscillations are active in the molecule. As has also been reported by Bonino and Cella, the characteristic frequency due to the carbon hydrogen linking has been obtained at 3046 cm.^{-1} , and that due to the aromatic linking $\text{C}=\text{C}$ has been obtained at 1582 and 1602 cm.^{-1} respectively, both of which appear as a doublet in the benzene spectrum and have been attributed by S. Bhagwantam * as due to a transverse oscillation of the hydrogen atom. The lines at wave-number shifts 1458 and 1433 cm.^{-1} obtained by the author, agree closely with those obtained by Bonino and Cella, and are due to 4358 Å. excitation. But the line at 1338 cm.^{-1} obtained and recorded by them as due to 4358 Å. excitation does not appear at all on the author's *m*-dinitrobenzene in benzene filter plate although it appears on his quinine sulphate filter plate, indicating that it is due to 4046 Å. excitation and not due to 4358 Å. excitation, as recorded by them. The line at wave-number 21,777 appears on both of the plates, appearing only weakly on the second plate, and therefore it is due to both 4358 Å. and 4046 Å. excitations. The lines at wave-numbers 23,329, 23,668, and 23,978 are very strong lines, and they also appear on both of the plates, but they are comparatively much weaker

* S. Bhagwantam, Ind. Jour. Phys. 5. p. 615 (1930).

on the second plate. Therefore they are all due to 4046 \AA . excitation, as has been reported by Bonino and Cella.

Amongst the other lines obtained are those due to the C-H groups and the aromatic ring in the molecule, and they correspond to all the lines which are due to the naphthalene molecule, except the one at the wave-number shift 938 (0), which is not obtained by the author but which is reported by Bonino and Cella. All these results will be found in Table VI. It will be seen from this table that a new frequency 2940 cm.^{-1} , not recorded by Bonino and Cella, has been obtained by the author. This is characteristic of the carbon hydrogen linking in the $:\text{CH}_2$ group and coincides remarkably well with 2938 cm.^{-1} obtained by P. Krishnamurti in the cyclo-hexane spectrum. There is a corresponding line in benzene at 2948 cm.^{-1} , which is usually attributed to hydrogen oscillations owing to its relatively low atomic weight.

The frequency obtained at 1037 cm.^{-1} requires special mention. It is shown as a very strong line on Pl. XX. (a), and it coincides very closely with the value obtained for it by Bonino and Cella. In cyclo-hexane as reported by P. Krishnamurti it appears with almost equal intensity. It also appears weakly in the dekalin and naphthalene spectra. It is present in the Raman spectra of all benzene derivatives. But, curiously enough, it is not present in the Raman spectrum of benzene, although there appears a strong infra-red absorption at 1030 wave-numbers in benzene. But there can be no doubt that it is characteristic of benzene and that the existence of the aromatic ring in the tetralin molecule is primarily the cause of its presence so strongly in its spectrum.

Another new frequency has been obtained by the author at 699 cm.^{-1} . This corresponds to 695 cm.^{-1} in cyclo-hexane, as reported by P. Krishnamurti. This line is also found to be partly excited by 4046 \AA ., as it appears much more weakly on the author's *m*-dinitro benzene filter plate than on his quinine sulphate filter plate. Bhagwantam * has reported one very weak line due to benzene at 694 cm.^{-1} , which closely agrees with 699 cm.^{-1} due to tetralin, and which, as reported by him, agrees remarkably well with a strong infra-red absorption found by Coblentz † at 694 wave-numbers.

* *Loc. cit.*

† Coblentz, Investigation of Infra-red spectra, Carnegie Inst. (1905).

He has not been able to explain its existence, although he suggests that owing to its high intensity of absorption it is a strongly active oscillation and must represent a very unsymmetrical mode. It might be suggested

TABLE VI.

Cyclo-hexane*.	Cyclo-hexane†.	Naphthalene ‡.	Dekalin (Mukerji).	Dekalin (Bonino & Cella)§.	Tetralin (Mukerji).	Tetralin (Bonino & Cella).
—	—	3046 (1)	—	—	{ 3046 (5) 2940 (4)	3049 (5)
2938 (8)	—	—	2922 (5)	2922 (4)	2918 (4)	2922 (4)
2922 (8)	2930 (3)	—	2892 (1)	—	—	—
2889 (1)	—	—	2855 (4)	2856 (4)	2865 (4)	2862 (2½)
2852 (8)	2848 (3)	—	—	—	—	—
2662 (1)	—	—	2658 (0)	—	1602 (3)	1602 (3)
—	—	1573 (3)	—	—	1582 (3)	1583 (3)
1444 (5)	1446	—	1447 (5)	1448 (5)	1458 (3)	1456 (3)
—	—	—	—	—	1433 (5)	1433 (3)
—	—	1412 (0)	—	—	—	—
1344 (½)	1343 (2)	1377 (4)	1362 (1)	1360 (2)	1376 (4)	{ 1374 (4½) 1340 (1½)
1266 (5)	1268 (3)	1245 (0)	1256 (3)	1260 (4)	{ 1283 (1) 1205 (6)	1283 (2) 1204 (4)
1156 (1)	1157 (3)	1142 (0)	1166 (3)	1165 (1)	1174 (0)	1160 (1)
1028 (8)	1028 (2)	1023 (2)	1024 (1)	{ 1046 (3½) 1017	1037 (7)	1038 (5)
—	992 (2)	936 (0)	991 (2)	990 (1)	?	938 (0)
—	—	—	851 (4)	851 (2½)	839 (0)	848 (0)
804 (10)	800 (4)	—	803 (3)	804 (1½)	814 (2)	815 (1½)
—	—	—	763 (5)	753 (3)	{ 752 (1) 723 (5)	759 (1) 724 (4)
695 (0)	—	—	—	—	699 (2)	—
—	608 (0)	610 (0)	596 (3)	593 (2½)	582 (3)	585 (3)
—	—	—	—	554 (2)	561 (2)	567 (2)
—	—	505 (2)	494 (3)	490 (2)	511 (3)	513 (3)
425 (1)	420 (2)	—	443 (1)	—	430 (4)	432 (2)
—	—	—	407 (3)	402 (2)	—	—
381 (½)	383	—	376 (1)	—	—	—
—	—	—	—	—	265 (2)	265 (3)
—	—	—	—	—	162 (3)	158 (4)

* P. Krishnamurti, Ind. Journ. Phys. vi. p. 543 (1932).

† Ganesan and Venkateswaren, Ind. Journ. Phys. iv. p. 195 (1929).

‡ R. Bär, 'Nature,' cxxiv. p. 692 (1929).

§ *Loc. cit.*

|| In a preliminary report to 'Nature,' cxxxiv. p. 811 (24th Nov. 1934), on the Raman Spectra of Dekahydro and Tetrahydro-naphthalene, the line at frequency shift 1283 cm.⁻⁴ due to tetralin, has been erroneously given as being also present in the naphthalene spectrum.

then that some similar mode of oscillation is also active in the tetralin molecule.

Tetralin has also given another line at 430 cm.⁻¹, which has also been reported by Bonino and Cella. This

line corresponds to 425 cm.^{-1} , due to cyclo-hexane as obtained by P. Krishnamurti. It may be noticed from Table VI. that the cyclo-hexane frequencies obtained by P. Krishnamurti coincide more closely with the corresponding tetralin frequencies obtained by the author than those obtained by Ganesan and Venkateswaren.

TABLE VII.

Dekalin with *m*-dinitrobenzene in benzene filter.

Number.	Wave-length of Raman line in I.A.	Frequency.	$\Delta\nu$.	Intensity.	Assignment.
1....	5074.12	19702	3236	0	A
2....	5058.91	19761	3177	0	A
3....	4994.62	20016	2922	5	A
4....	4987.0	20046	2892	1	A
5....	4978.02	20083	2855	4	A
6....	4929.50	20280	2658	0	A
7....	4651.8	21491	1447	5	A
8....	4633.63	21576	1362	1	A
9....	4630.68	21588	1350	1	A
10....	4614.35	21665	1273	1	A
11....	4610.67	21682	1256	3	A
12....	4591.82	21772	1166	3	A
13....	4576.3	21846	1092	3	A
			2859	5	B
14....	4568.84	21881	1057	2	A
			2824	2	B
15....	4566.95	21890	1048	3	A
16....	4562.1	21914	1024	1	A
17....	4555.08	21947	991	2	A
18....	4543.06	22005	933	2	A
19....	4532.21	22058	880	1	A
20....	4526.25	22087	851	4	A
21....	4516.51	22135	803	3	A
22....	4505.8	22187	763	5	A
23....	4474.65	22342	596	3	A
24....	4454.26	22444	494	3	A
25....	4444.12	22495	443	1	A
26....	4437.02	22531	407	3	A
27....	4430.92	22562	376	1	A
28....	4426.18	22586	352	1	A
29....	4420.71	22614	324	0	A
30....	4414.92	22644	294	0	A
31....	4391.51	22765	173	0	A ₁
32....	4387.73	22784	156	0	A ₁
33....	4331.08	23082	1623	00	B
34....	4328.07	23098	1607	00	B
35....	4298.51	23257	1448	2	B
36....	4282.3	23345	407	2	A
37....	4266.32	23432	494	1	A
38....	4247.41	23534	596	0	A

TABLE VIII.
Tetralin with quinine sulphate filter.

Number.	Wave-length of Raman line in I.A.	Frequency ν in cm.^{-1} .	A.D. 4ν .	Intensity.	Assignment.
1....	5025.6	19892	3046	5	A
2....	4998.85	19998	2940	4	A
3....	4993.4	20020	2918	4	A
4....	4980.38	20073	2865	4	A
5....	4685.6	21336	1602	3	A
6....	4681.05	21356	1582	3	A
7....	4653.95	21480	1458	3	A
8....	4648.65	21505	1433	5	A
9....	4636.35	21562	1376	6	A
10....	4630.3	21591	3114	1	B
11....	4617.95	21648	3057	2	B
12....	4617.95	21648	1290	1	A
13....	4604.6	21711	1283	1	A ₁
14....	4599.5	21735	1205	5	A
15....	4594.78	21757	2948	5	B
16....	4590.65	21777	{ 2928 1161 }	2 1	B A
17....	4578.8	21833	1105	0	A
18....	4572.27	21864	1074	0	A
19....	4564.7	21901	1037	6	A
20....	4553.26	21956	1038	1	A ₁
21....	4523.7	22099	839	0	A
22....	4518.72	22124	814	2	A
23....	4508.1	22176	752	2	A
24....	4500.0	22215	723	5	A
25....	4495.25	22239	{ 2466 699 }	2 0	B A
26....	4471.77	22356	582	3	A
27....	4457.57	22427	511	3	A
28....	4441.55	22508	430	4	A
29....	4409.41	22673	265	2	A
30....	4401.2	22715	223	0	A
31....	4389.37	22776	162	3	A
32....	4327.7	23100	162	1	A
33....	4308.63	23203	265	0	A
34....	4298.4	23257	1448	1	B
35....	4285.25	23329	1376	4	B
36....	4278.8	23364	1341	2	B
37....	4253.5	23503	1202	4	B
38....	4248.4	23531	1174	0	B
39....	4223.97	23668	1037	6	B
40....	4197.9	23814	891	2	B
41....	4186.6	23879	826	0	B
42....	4175.3	23943	762	1	B
43....	4169.2	23978	727	5	B
44....	4165.3	24001	704	1	B
45....	4143.6	24126	579	3	B
46....	4140.6	24144	561	2	B
47....	4131.4	24198	507	3	B
48....	4119.0	24270	435	3	B
49....	4102.1	24371	1433	1	A
50....	4089.3	24446	259	2	B

TABLE IX.

Tetralin with *m*-dinitro-benzene in benzene filter.

Number.	Wave-length of Raman line in I.A.	Frequency ν .	$\Delta\nu$.	Intensity.	Assignment.
1....	5025.6	19892	3046	4	A
2....	4998.85	19998	2940	3	A
3....	4993.4	20020	2918	2	A
4....	4980.38	20073	2865	2	A
5....	4685.6	21336	1602	2	A
6....	4681.05	21356	1582	2	A
7....	4653.95	21480	1458	2	A
8....	4648.65	21505	1433	3	A
9....	4636.35	21562	1376	4	A
10....	4617.95	21648	1290	1	A
11....	4604.6	21711	1283	1	A ₂
12....	4599.5	21735	1205	4	A
13....	4590.65	21777	1161	1	A
14....	4578.8	21833	1105	0	A
15....	4572.27	21864	1074	0	A
16....	4564.7	21901	1037	4	A
17....	4553.26	21956	1038	0	A ₁
18....	4523.7	22099	839	0	A
19....	4518.72	22124	814	2	A
20....	4508.1	22176	752	2	A
21....	4500.0	22215	723	4	A
22....	4495.25	22239	699	0	A
23....	4471.77	22356	582	2	A
26....	4457.57	22427	511	2	A
25....	4441.55	22508	430	3	A
26....	4409.41	22673	265	1½	A
27....	4401.2	22715	223	0	A
28....	4 89.37	22776	162	3	A
29....	4327.7	23100	162	½	A
30....	4308.63	23203	265	0	A
31....	4285.25	23329	1376	2	B
32....	4223.97	23668	1037	2	B
33....	4169.2	23928	727	2	B
34....	4102.1	24371	1433	00	A

TABLE X
Exciting lines.

Symbol used.	Wave-length in I.A.	Wave-number.	Intensity.
A	4358.6	22938.1	10,000
A ₁	4347.5	22994	600
B	4046.8	24705	7000

In conclusion the author wishes to thank Prof. Sir C. V. Raman, F.R.S., in whose laboratory at Calcutta this work was carried on until he left Calcutta early in January 1933, for kindly placing at his disposal the facilities of his laboratory; Prof. O. W. Richardson, F.R.S., in whose laboratory at King's College, London, this work was completed, for kindly permitting him to use the micro-photometer; and Dr. W. E. Williams for kindly taking interest in this work and for permission to use the comparator. Thanks are also due to Mr. H. Krall, Professor of Chemistry, Agra College, for kindly supplying the two liquids investigated.

Summary.

Dekahydro and tetrahydro naphthalene have been investigated for their Raman spectra with a view to recording the faint lines and assigning the frequencies correctly.

Dekalin has yielded seventeen new lines not recorded before at wave-numbers 19,702, 19,761, 20,046, 21,588, 21,665, 21,881, 22,005, 22,495, 22,562, 22,586, 22,614, 22,644, 22,765, 22,784, 23,082, 23,098, and 23,173. Tetralin shows twelve new unrecorded lines at wave-numbers 19,892, 19,998, 21,833, 21,864, 22,239, 22,715, 23,100, 23,203, 23,814, 24,001, 24,371, and 24,446.

They have each given three anti-Stokes lines not recorded before at frequencies 596, 494, 407, and 1433, 265, and 162 cm.^{-1} respectively.

In addition, nearly all the other lines obtained by G. B. Bonino and P. Cella for dekalin and tetralin have been obtained. The author's results have been compared with those of G. B. Bonino and P. Cella, and also with the naphthalene and cyclo-hexane spectra and are discussed at length.

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XCII. *On Complex Primes.*

By LUDWIK SILBERSTEIN, *Ph.D.* *

THE so-called Theory of Numbers, one of the most fascinating branches of mathematics, has since the time of Euclid concerned itself (apart from a very

* Communicated by the Author.

few exceptions) with the properties of *real* integers, the primality of a given real integer and the co-primality (or its negation) of two such numbers being the central questions.

The discrete manifold studied in the Theory of Numbers was, then, one-dimensional or, as we may say, a *linear* manifold of elements, 1, 2, 3, etc.

Originally the Theory of Functions, say of a single independent variable, as, e. g., $f(x) = \log x$ or $\sin x$, or $\tan x$, etc., was also limited to the real domain, both x and f being one-dimensional continua.

Now, it is generally known how fruitful the extension of these fields (of x and f) to the *complex domain* has turned out to be, not only leading to new and interesting properties of functions, but throwing a flood of light upon their already known properties within the narrower, real domain.

By analogy, then, one was naturally led to expect some advantages from the extension of the real domain of the usual Theory of Numbers to the complex domain. This has (some months ago) suggested the idea of systematically investigating the properties of *complex integers*, i. e., of numbers

$$z = x + iy,$$

where $i = \sqrt{-1}$ and x, y are ordinary real integers*.

The material of the Theory of Numbers thus broadened forms, then, a (discrete) *bidimensional* manifold of elements, as, e. g., $1 + i, 1 + 2i, 1 + 3i; 2 + i, 2 + 3i$, etc., of which the usual integers, 1, 2, 3, etc., constitute but a subclass.

The first question which presented itself was naturally that of *primality*.

The real numbers x, y being given integers, is the complex number $z = x + iy$ a product of two or more other

* As I had at that time (June 1934) ready access only to Kronecker's excellent 'Vorlesungen über Zahlentheorie,' vol. i. (edited by Hensel), where there is no mention whatever of complex integers, I was under the impression that this extension is new. Later (Sept. 1934), consulting the Enc. Sc. Math., I found that complex integers, or equivalently pairs of real integers, were considered and extensively studied by Felix Klein in his 1895-96 lectures, 'Ausgewählte Kapitel der Zahlentheorie,' edited by A. Sommerfeld (autographed), Göttingen, 1896. But (cf. *infra*) his concept of "prime lattice points" (Prim-punkte) has turned out to differ from that of the "complex primes" as here defined. Also, the method adopted by Klein would seem unnecessarily cumbersome.

complex integers or not? In other words, has z complex divisors other than 1 and itself or not? In the latter case it will be called a *complex prime* or, more briefly, a z -prime.

The first question naturally to suggest itself was: Which of the familiar "primes" remain z -primes, and which are to be erased or, so to speak, fall under the complex-guillotine?

Manifestly such, and only such, real primes will be products of complex integers which are sums of two squares, as, *e. g.*,

$$13 = 2^2 + 3^2 = (2 + 3i)(2 - 3i).$$

Similarly for $5 = 1^2 + 2^2$, and so on.

If P , a "real" prime (*i. e.*, prime in the one-fold of real integers) is to be a *composite* (non-prime) number, it must be of the form

$$P = (x + iy)(x - iy) \equiv zz' \text{ *},$$

i. e., $P = x^2 + y^2$, a sum of two squares.

It will be convenient to denote z -primes by p , and to reserve the symbol P for real primes, *i. e.*, those of the one-dimensional field.

Thus, no prime P of the form $x^2 + y^2$ is a z -prime; briefly,

$$P = x^2 + y^2 \neq p.$$

Now, as I have actually found empirically all primes (after 2) which are sums of two squares are of the form $4n + 1$, and conversely. I thought I had discovered a new property; but, after a protracted search (in Auerbach's Pocket-books), this property has turned out to have already been announced by the all-powerful Fermat, and then proved by Euler. At any rate, however, I have been led to this rediscovery (having certainly never heard of it before) by the extension of real to complex integers, which shows, as in the Theory of Functions, the heuristic value of such an extension; and it is mainly this circumstance which has encouraged me to spend some more time on the subject of this paper.

Hence, what from the series of ordinary primes is erased or cut out by the complex-guillotine are the primes of the form

$$P = 4n + 1,$$

* The conjugate of z will be throughout denoted by z' .

and, apart from $2=1^2+1^2$, *only these*. In fine,

$$2, 5, 13, 17, 29, \text{ etc.},$$

cease to be prime numbers, while *all*

$$P=4n+3,$$

and only these, remain prime in the extended field. Briefly, every

$$P=4n+3=p.$$

These, then, and only these, numbers on the real axis of the two-fold remain as primes, viz.,

$$3, 7, 11, 19, 23, 31, \text{ etc.}$$

Each of the abolished P 's $(4n+1)$ gives rise to two numbers, to a pair of conjugate complex integers, z and z' ,

$$P=4n+1=x^2+y^2=zz' = |z| |z'|.$$

Are z, z' themselves primes, *i. e.*, is always $z=p, z'=p'$? The answer is in the affirmative.

In fact, suppose that

$$z=r_1e^{\theta_1 i} \cdot r_2e^{\theta_2 i}=r_1r_2e^{(\theta_1+\theta_2)i},$$

where r_1^2 and r_2^2 are integers. Then $z'=r_1r_2e^{-(\theta_1+\theta_2)i}$ and $zz'=r_1^2r_2^2=a$ product of two integers, so that the number in question would not be a P , against the assumption.

Thus, $z=x+iy$ and $z'=x-i'y$, the two descendants, so to speak, of the guillotined P , are *primes*, p and p' .

The corresponding pair of points of the z -plane is symmetrical with respect to the x -axis.

But (with the only exception of $2=1^2+1^2$) every such erased real prime $P=4n+1$ gives rise to yet another pair of z -primes, namely,

$$\zeta=y+ix \quad \text{and} \quad \zeta'=y-ix;$$

the points ζ and z are symmetrical with respect to the bisectrix of the right angle YOX , and similarly ζ' and z' with respect to the bisectrix of $-YOX$.

Hence we have four prime points, z, z', ζ, ζ' , and as the reflexions of these in the y -axis ($-x+iy$, etc.) again satisfy the condition $P=x^2+y^2$, we have in all eight primes, as the descendants, so to speak, of the abolished prime P , the corresponding eight points being all distributed on a circle of radius $r=\sqrt{P}$ (cf. fig. 1).

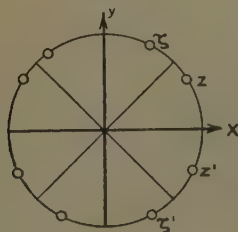
Thus, *e. g.*, the abolished prime $13=2^2+3^2$ is replaced by the eight complex primes

$$\begin{array}{cccc} 2+3i & 2-3i & 3+2i & 3-2i \\ -2+3i & -2-3i & -3+2i & -3-2i, \end{array}$$

the latter four differing from the former only by the factor -1 . Similarly for the descendants of all other abolished real primes, with the exception of 2, for which $z=\zeta=1+i(x=y)$.

To obtain a good idea of the distribution of primes in the z -plane it is enough to consider only the first and the fourth quadrants, *e. g.*, those of positive x . Using squared paper, I have represented the z -primes replacing the abolished real primes, as well as the surviving real primes of the form $4n+3$, by black unit squares (the

Fig. 1.



centre of each square corresponding to the z -value in question), in accordance with the numerical table appended at the end of the paper.

This graphical process gave rise to a beautiful pattern or mosaic of primes, a good portion of which is shown in fig. 2. The black squares centred on the horizontal (real) axis stand for the primes 3, 7, 11, 19, etc. ($4n+3$), while the remaining ones represent the complex offspring (or débris) of the abolished real primes of form $4n+1$.

The next question to present itself was: Are these offspring of the abolished primes *P the only existing z-primes?*

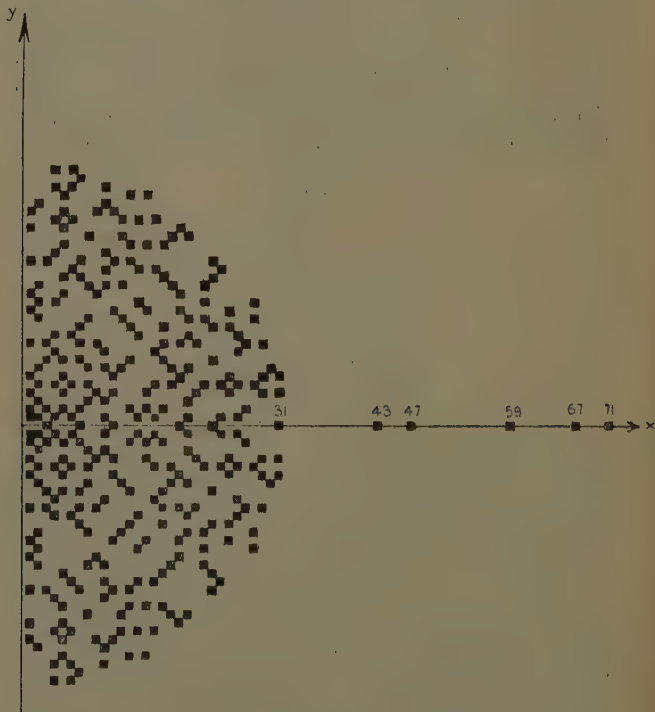
I feel confident that the answer is in the affirmative, though I have so far been unable to prove it analytically. My confidence is based partly upon extensive trials, *i. e.*, actual decomposition into factors of complex integers

$1+ni$, $2+ni$, etc., and partly upon a certain theorem proved by Klein for a different, yet somewhat analogous, situation.

First, as regards those trials. Let a given complex integer z be the product of two complex integer factors, say

$$x+iy=(\alpha+i\beta)(\gamma+i\delta).$$

Fig. 2.



The square $x=30$, $y=7$, and the three corresponding ones are superfluous.

Then

$$\gamma = \frac{x\alpha + y\beta}{\alpha^2 + \beta^2}, \quad \delta = \frac{y\alpha - x\beta}{\alpha^2 + \beta^2};$$

x and y being given integers, one has to find by systematic trial whether or not there are integral α, β which yield integers for γ and δ . The process is by no means prohibitively laborious. Thus, starting with $z=1+ni$, one readily finds that

$$1+3i=(1+i)(2+i), \quad 1+5i=(1+i)(3+2i),$$

$$1+7i=(1+i)(4+3i), \text{ etc.}, \quad 1+19i=(1+i)(10+9i),$$

$$1+21i=(1+i)(11+10i),$$

while $1\pm i, 1\pm 2i, \dots, 1\pm 20i$ are all débris of discarded P 's ($4n+1$). Similarly,

$$2+9i=(1+2i)(4+i), \quad 2+11i=(2+i)(3+4i),$$

$$2+19i=(1+2i)(8+3i), \quad 2+21i=(2+i)(5+8i),$$

while $2\pm 3i$, etc., $2\pm 17i$ are descendants of abolished P 's. Again,

$$3+i=(1+i)(2-i), \quad 3+4i=(2+i)^2, \dots$$

$$3+23i=(1+i)(13+10i),$$

while $3\pm 2i, 3\pm 8i, 3\pm 10i, 3\pm 20i$ are all débris of discarded P 's, and so on. I am still extending this set of trials, but so far all the complex integers (other than the said débris of primes $4n+1$) have turned out to be composite.

Second, Klein's theorem, alluded to above, and proved in his lectures (*loc. cit.* ii. p. 209), reads as follows:—

"All prime points of our figure [planar lattice] are obtained by decomposing into factors those points (p, p) situated upon the horizontal straight line of the principal lattice, whose coordinates p are prime in the sense of the usual Theory of Numbers."

Klein's concept of "unit point" (Einheitspunkt, *ibid.*, p. 175) differs from our, the ordinary, unit; nay, his "unit points" form a plurality, a finite or infinite discrete manifold. Thus, his "prime points" do by no means coincide with the graphical representatives of our complex primes $x+iy$. (*Cf.*, e. g., pp. 242-3, ii. *loc. cit.*, where 2 and 5 appear as Kleinian primes, while from our standpoint they are composite, and $7=(2+\sqrt{-3})(2-\sqrt{-3})$ is, for Klein, composite, whereas in our treatment it is a z -prime, $4.1+3$).

The theorem, whose correctness we confidently expect, can be stated thus :—

If $z = x + iy$ is a prime, then *

$$zz' = |z|^2 = P = 4n + 1,$$

i. e.,

$$x^2 + y^2 = P = 4n + 1.$$

Perhaps some of my readers will be able to give a general proof of this theorem, which, in the meantime, we must accept here without proof.

Under these circumstances the primes obtained above from the decomposition of the ordinary primes of form $4n + 1$, and marked by black squares in fig. 2, constitute, with the ordinary $(4n + 3)$ -primes, the totality of z -primes. Our *diagram*, that is, gives the distribution of all z -primes contained (in the first and the fourth quadrant) within a circle of a certain radius r , in the case of fig. 2, actually

$$r = \sqrt{15^2 + 28^2} = \sqrt{1009}.$$

I cannot help repeating that the æsthetic, and certainly endless, variety of groupings in this pattern, the distribution of the z -primes, has greatly impressed me. The “irregularity” of distribution of the ordinary, real primes, generally known, is still enhanced here, in the two-dimensional manifold of complex integers.

The discussion of several characteristic peculiarities of this distribution of complex primes, readily perceivable on our graphical pattern, may be relegated to a future paper.

APPENDIX.

The preceding table gives the decomposition of the (abolished) “primes” of form

$$P = 4n + 1 = x^2 + y^2 = (x + iy)(x - iy)$$

up to one thousand.

I have extended this decomposition of $(4n + 1)$ -primes a good deal further, but there is, in the present connexion, no reason for occupying any more space in this journal.

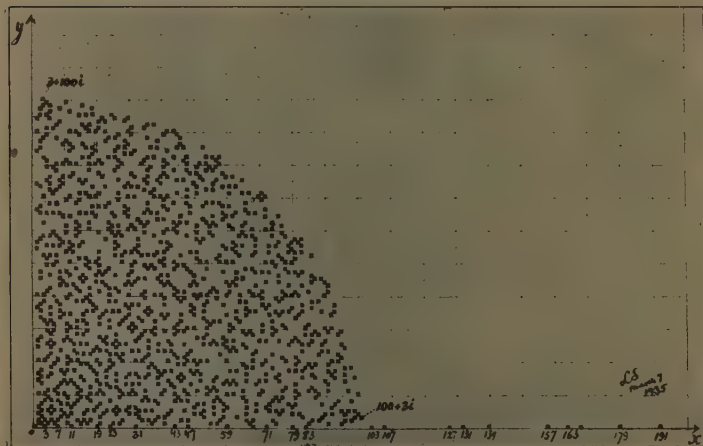
Rochester, N.Y.,

January 8, 1935.

* Apart from $z = 1 + i$, for which $zz' = 2$.

Note added in proof, March 21.—Since this article was sent to the Editors I have considerably extended the table of $P=4n+1$, x, y , viz., up to $P=25,033$. Part of the material up to $P=10,009$, is graphically represented

Fig. 3.



where l, m are real integers, so that we could write

$$z = \bar{l}e^{i(\theta-\alpha)} \cdot \sqrt{m}e^{i\alpha}. \quad (1)$$

$$\text{Now, } \tan(\theta-\alpha) = \frac{\tan \theta - \tan \alpha}{1 + \tan \theta \cdot \tan \alpha}.$$

Since z is, at any rate, a complex *integer*, $\tan \theta$ is a rational number, viz., the ratio $x:y$. Thus, we have only to choose $\tan \alpha$ rational (e. g., $\alpha = \pi/4$, $\tan \alpha = 1$) to make also $\tan(\theta-\alpha)$ rational. Thus, by (1), z would be a product of two complex integers, against the assumption.

Hence $|z|^2$ is a real "prime," and since it is a sum of two squares, $x^2 + y^2$, it is, by Euler's theorem, of the form $4n+1$. Q.E.D.

Thus the ordinary primes of the form $4n+3$ and the "offspring" of the abolished "primes" of form $4n+1 = x^2 + y^2 = (x+iy)(x-iy)$ make up the *totality* of primes of the complex domain, as expected in the paper.

Rochester, N.Y.

March 28, 1935.

XCIH. *The Personal Error in Colorimetry.* By R. A. HOUSTOUN, D.Sc., F.Inst.P., and ALEXANDER J. YOUNGER, M.A., B.Sc., Lecturers in the University of Glasgow*.

§1. **A**S is well known the laws of colour mixing can be represented on a diagram dating from Newton's 'Opticks,' in which the colours of lights are represented by the positions of points and their quantities by the masses of particles placed at these points. The result of mixing two colours then follows from the process of finding the centroid of the two particles. Different systems of coordinates are possible, but the Commission Internationale de l'Eclairage held at Cambridge in 1931 authorized a special system for use in colorimetry.

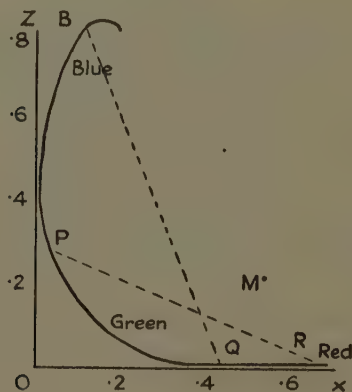
In a recent article in this Magazine † Houston described an arrangement by which accurate colorimetry can be done with a spectrometer. This employs the vector

* Communicated by the Authors.

† Phil. Mag. xvii. p. 1047 (1934).

method of colorimetry, and can be illustrated by fig. 1. In this diagram the position of the colours of the spectrum is given by the curve. P is a point in the blue-green; if a red light is superposed on this part of the spectrum and the relative proportion of the two colours altered the colour of the mixture travels along the line RP. Similarly, if a blue light is superposed on the colour Q in the yellow of the spectrum and the relative proportion of the two colours altered the colour of the mixture travels along the line BQ. The process of matching a colour consists in finding two straight lines on which it

Fig. 1.



lies. Its coordinates are then given by the point of intersection of these straight lines. For details of the method reference should be made to the original paper.

The Commission already referred to sanctioned three standard illuminants for use in colorimetry, a gas-filled lamp operated at the colour temperature of 2848°K. , and the same lamp in combination with liquid filters to yield sources of colour temperature of about 4800°K. and 6500°K. In resolution (2) of the Commission these standards are referred to as A, B, and C. Their coordinates should be, by calculation, $(.448, .145)$, $(.348, .300)$, and $(.310, .374)$; each colour has three coordinates x, y, z , but as $x+y+z=1$ always, one of these coordinates is superfluous, and in this paper it is always the x and z

coordinates that are referred to. These standards form valuable tests for the observer. It is stated in the previous paper that Houstoun measured them, and obtained as results ($\cdot 469$, $\cdot 092$), ($\cdot 379$, $\cdot 209$), and ($\cdot 349$, $\cdot 266$). His results are thus in marked disagreement with the C.I.E. standard observer. On placing the Wratten "tungsten to daylight" filter, no. 78, in front of the A lamp, and on placing the Wratten magenta filter, no. 32, in front of the A lamp, Houstoun obtained ($\cdot 354$, $\cdot 248$) and ($\cdot 617$, $\cdot 122$) respectively.

These same five test colours were thereupon measured by A. J. Younger. His values for A, B, and C were respectively ($\cdot 452$, $\cdot 136$), ($\cdot 348$, $\cdot 300$), and ($\cdot 292$, $\cdot 368$). They are thus in good agreement with the C.I.E. observer. For A with the tungsten to daylight filter Younger obtained ($\cdot 342$, $\cdot 307$). The best line to get on magenta is the direct one from red to blue. Its intersection with the lines through the above-mentioned four points were found by Younger to be ($\cdot 522$, $\cdot 233$), ($\cdot 522$, $\cdot 230$), ($\cdot 554$, $\cdot 196$), and ($\cdot 478$, $\cdot 292$), giving a mean for magenta of ($\cdot 519$, $\cdot 238$).

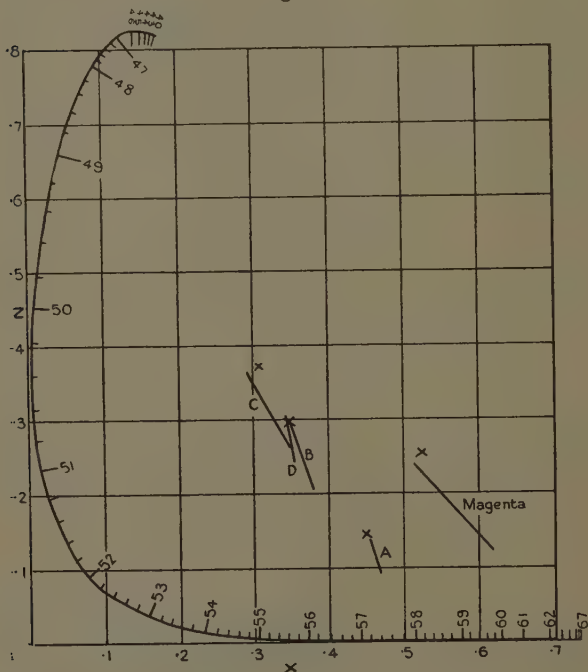
In fig. 2 are entered Houstoun's and Younger's readings for lamps A, B, C, A with the daylight filter (denoted by D), and A with magenta, the two readings for each source being joined by a straight line. The positions of the C.I.E. observer are indicated by crosses. In one case he coincides with Younger; in the other cases his difference with Younger is slight, as has already been mentioned. The difference between Houstoun and Younger is always in the same sense.

§ 2. As neither of the observers has any abnormality of colour vision it was at first not known how to explain the difference. W. D. Wright* got forty-six observers to match "white" with a trichromatic colorimeter, and disregarded the results of ten, "three of them being in some way anomalous, and seven showing marked inaccuracy and unreliability." The results of the remaining thirty-six are plotted on a colour-diagram. Wright does not employ the C.I.E. system of coordinates, as his paper was written before this system was established, so a direct comparison between his diagram and ours is not possible.

* A Re-determination of the Trichromatic Coefficients of the Spectral Colours," *Trans. Opt. Soc.* xxx. p. 151 (1928-29).

Some of his observers appear to show bigger differences than we do, and he states that the variations are "not insignificant quantities, but are surprisingly big. They must, in fact, seriously affect any system of colour measurement which does not make allowance for such differences."

Fig. 2.



Unfortunately Wright gives his results only by means of the diagram, which does not print well, nor does he explain in detail why he rejected ten out of forty-six observers. We should have expected only about two seriously colour-blind in this number. As the differences between the actual observer and the C.I.E. observer are of critical importance for colorimetry, we thought it desirable to get an idea of their magnitude from another quarter.

The visibility curve varies from observer to observer. The visibility curve of the C.I.E. observer could be made to agree with that of any other observer by placing a suitably coloured glass filter in front of his eye. Now the coordinates of a source the energy of which does not vary with the wave-length are ($\cdot 333$, $\cdot 333$) according to the C.I.E. observer. The question arises as to what they would be if he viewed it through a filter which would convert his visibility curve to one of the other forms occurring amongst observers of normal colour vision. In comparing the C.I.E. observer with the actual observer there are differences due to the different visibility curve of the latter and to his different colour vision. If we find what effect the first of these factors has on the equal energy point we shall have an idea of the differences existing amongst actual observers, probably an underestimate, because we are considering in this way only one of the factors that cause difference.

The most useful data for a calculation of this nature are those of Coblentz and Emerson *. They investigated 130 subjects, 7 of which were colour-blind. The remaining 123 subjects, who had normal colour vision, were divided into seven classes according to the shape of their visibility curves. The number in each class, the name of the class, and the abbreviation for the name are given in the following table :—

Average	N.	29
Wide	W.	7
Red sensitive	R.S.	26
Blue sensitive.....	B.S.	20
Green sensitive	G.S.	13
Subnormal red	Sub.R.	17
Subnormal blue.....	Sub.B.	11

The classification is obviously obtained by dividing the spectrum into three parts, and considering whether one or two of these parts is below the normal.

The observers made settings at 35 points in the spectrum, but few of them made settings at all the 35 points. We took as representative of each class those observers in the class who made most settings, and obtained

* W. W. Coblentz and W. B. Emerson, Bull. Bur. of Standards, xiv. p. 167 (1918).

1112 Dr. R. A. Houstoun *and* Mr. A. J. Younger :

the following table, the maximum visibility in each case being taken about 1000. The N column is the mean of Coblentz and Emerson's observers, nos. 10, 62, and 75, W. is the mean of nos. 1 and 61, R.S. the mean of nos. 43

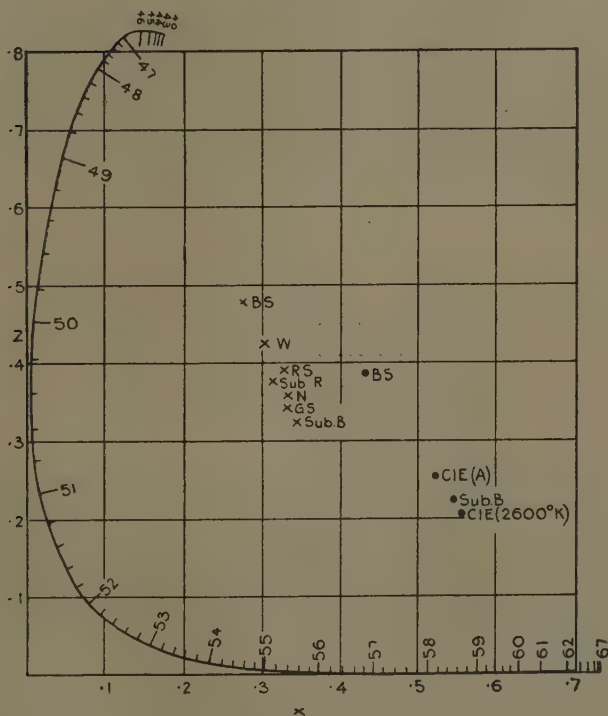
λ .	N.	W.	R.S.	B.S.	G.S.	Sub.R.	Sub.B.
m μ .							
427	23.3	28.5	—	38.0	—	—	—
435	28.8	40.6	36.0	51.2	24.7	27.3	—
444	36.4	50.6	44.2	64.4	25.9	35.2	26.4
450	42.4	61.0	50.8	74.0	27.5	39.9	29.5
456	55.7	74.0	64.4	93.5	32.5	51.0	33.5
463	69.4	102.5	76.3	121.5	52.9	63.5	44.6
474	106.4	147.5	106.6	173.0	77.3	109.0	71.9
485	164.0	227.0	164.5	251.5	130.0	156.0	109.9
493	236.0	295.0	243	337	176.0	218.0	168.0
502	373.0	430	376	476.0	310.0	368	315
512.5	556	607	545	640.0	538	560	509
523	760	772	733	816.0	777	780	752
534	886	893	883	924.0	920	920	909
546	970	959	956	968.0	982	963	996
552	981	992	970	988.0	998	992	1003
559	991	1000	1000	993.0	1000	1008	982
573	939	972	977	933	940	925	936
580	873	936	973	874	873	865	899
587.6	806	863	903	795	783	768	823
596.4	726	769	818	708	711	679	743
604	623	686	740	611	601	578	632
613	510	535	597	498	467	429	521
623	384	411	465	380	340	327	404
633	270	288	340	261	231	211	281
643	171.0	184.0	217.0	160.5	146	132.5	177.3
654	93.3	100.4	118.0	89.4	78.6	75.0	96.8
665	46.3	50.2	61.7	45.0	38.2	37.8	48.0
678	21.0	23.9	27.6	20.9	17.2	16.7	22.0
690	8.93	9.83	11.45	9.35	10.2	6.82	9.31
703	3.91	4.42	4.96	4.11	3.22	2.99	4.01
717	1.54	1.62	1.81	1.54	1.21	1.16	1.49
730	.546	.666	.719	.579	.528	.457	.620
746	.279	.314	.331	.335	.287	.212	.305

and 63, B.S. the mean of nos. 46 and 117, G.S. is no. 44, Sub.R. the mean of nos. 8 and 65, and Sub.B. the mean of nos. 55, 58, and 64. The fact that some of the numbers are above 1000 is due to Coblentz and Emerson having

only approximately adjusted the individual curves to the same maximum.

The above tables were graphed, the ordinates read off at $5m\mu$ intervals, supplemented at the ends of the range by the values for the C.I.E. observer, and the position

Fig. 3.



	x .	z .
N.3334	.3581
W.3039	.4284
R.S.3286	.3901
B.S.2789	.4799
G.S.3320	.3474
Sub.R.3149	.3784
Sub.B.3486	.3259

of the equal-energy source calculated for each of the seven classes *. The results are given in the preceding table and shown by crosses in fig. 3. The curve in the figure gives the locus of the spectrum.

We then proceeded to calculate the position of the magenta filter, the Wratten standard complementary no. 32, used with the standard lamp A, as seen by the C.I.E. observer, and also as seen by our B.S. and Sub.B. observers. For this purpose we used the transmission factors of the filter supplied by the makers. Finally, we calculated the position of the same filter placed in front of a lamp of colour temperature 2600°K. , as seen by the C.I.E. observer. Lamp A has, it will be remembered, a colour temperature of 2848°K. The results are given in the following table, and shown as disks in fig. 3.

Position of Magenta.

	$w.$	$z.$
C.I.E. (A)	·5236	·2538
B.S. (A)	·4357	·3880
Sub.B. (A)	·5482	·2235
C.I.E. (2600°K.) ..	·5577	·2047

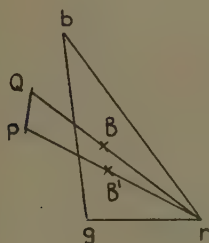
The scatter of the points in fig. 3 is much greater than we anticipated. It is obvious that the difference between Houston and Younger can be explained on the assumption that the former is subnormal blue, also that accidental variation of the current through the lamp is much less serious as a source of error than the personal error of the observer.

§3. The complete C.I.E. system requires that all observers using it shall have identical readings for the wave-lengths $700\text{ m}\mu$, $546\cdot1\text{ m}\mu$, $435\cdot8\text{ m}\mu$, and the standard illuminant B. In the case of a trichromatic colorimeter different observers are brought into agreement at the point B by assigning to them different luminosity factors for the red, green, and blue primaries of the instrument. Undoubtedly this is the best way to treat the difficulty; but

* *I. e.*, the ordinates were substituted for the standard values of y , and x and z calculated from the ratios x/y and z/y .

it throws the observers out of agreement elsewhere. For example, in fig. 4 r , g , b , and B are the four points on which observers are to be in agreement, and PQ is a portion of the spectrum curve according to the C.I.E. observer. If the position of PQ be assumed, an observer with Houstoun's vision finds the colour represented by B to be at B' . Suppose it shifted to B : the line from r through it cuts the spectrum curve now in Q instead of P , at $496\text{ m}\mu$ instead of $501\text{ m}\mu$, the z coordinates of which points are $\cdot542$ and $\cdot430$. Now the observer knows that his complementary to r is $501\text{ m}\mu$. He thus obtains the z coordinate of this wave-length wrong by the difference of these numbers, $\cdot112$, quite a large error.

Fig 4.



There are three different methods of using the C.I.E. system of colorimetry—indirect colorimetry, in which the reflectance of the specimen is measured by a spectrophotometer wave-length by wave-length; direct colorimetry with a “robot” observer, *e. g.*, three photo-cells provided with appropriate filters; and direct colorimetry with the human observer. The errors to which the first two methods lie open are obvious and should not lead any one wrong; but if the third method is used by an observer differing from the C.I.E. observer the errors are more insidious, and we are afraid they will require to be guarded against by a careful selection of observers. For if in a case in the law courts we have a number of observers obtaining different numerical values for the same colour, the system of measurement will suffer a serious loss of credit.

XCIV. *The Raman Spectra of Amides and Annelides in the Fused State.*—Part I. By V. N. THATTE and M. S. JOGLEKAR, *M.Sc., Physics Department, College of Science, Nagpur* *.

[Plate XXI.]

WHILE the Raman spectra of amines† has been studied in detail the amides and annelides have not received much attention, partly because the majority of these compounds are solid at room temperature and also because the higher members of these two groups of compounds are sparingly soluble in water. In the present paper the Raman spectra of some amides and annelides in the fused state have been studied. The chemicals were supplied by Kahlbaum and were contained in a Wood's tube placed axially in a specially constructed electric furnace. The substance was maintained at a temperature a few degrees above the melting-point. The liquid was illuminated by a quartz mercury lamp, light being admitted through a side window. The spectra of the scattered light were photographed by means of a Hilger E₃ quartz spectrograph.

The results of measurements of the modified lines and the corresponding shifts in the spectra of the scattered light are given in Tables I. to VI. The intensities of the modified lines are estimated visually.

The spectrum contains three very strong lines, which are also broad (see Table I.). The corresponding shifts are $\Delta\nu=1388$, 1309, 1112. Formic acid, of which formamide is a derivative, gives also a strong shift at $\Delta\nu=1390$. The shift at $\Delta\nu=1675$ is due to the C=O group, and this frequency is less than that found in acids and ketones. This lowering of the value of the frequency is due to the presence of the NH₂ group.

The shift at $\Delta\nu=870$ is very prominent and sharp. This shift also occurs in the spectra of acetic acid and its derivatives, and is attributed to the C-C linkage in the aliphatic acids. Corresponding to the strong shift $\Delta\nu=1388$ in formamide we have here a strong shift

* Communicated by the Authors.

† Venkateswaran and Bhagwantam, *Ind. Journ. of Physics*, v. p. 129 (1930; also Dadiou and Kohlrausch, *Sitz. Akad. Wiss. Wien*, cxxxix. p. 459 (1930).

TABLE I.

Formamide (liquid at room-temperature).



Modified lines.		Wave-number (cm. ⁻¹).	Exciting line.	Wave-number difference, $\Delta\nu$.
Wave-length in vacuum.	Intensity.			
4703.0	(0)	21263	4358.3	1675
			(22938 cm. ⁻¹)	
4680.3	(1)	21366	4046.6	3339
			(24705 cm. ⁻¹)	
4640.3	(2 broad)	21550	4358.3	1388
4623.4	(2 very broad)	21629	"	1309
4581.6	(2)	21826	4358.3 } 4046.6 }	→1112 →2879
4502.1	(1)	22212	4358.3	726
4477.2	(0)	22335	"	603
4435.7	(0)	22544	"	394
4286.4	(0)	23330	"	-392
4273.6	(1)	23399	4046.6	1306
4238.0	(0)	23596	"	1109
4149.3	(0)	24100	"	605

TABLE II.

Acetamide (melting-point 83° C.).



Modified lines.		Wave-number (cm. ⁻¹).	Exciting line.	Wave-number difference, $\Delta\nu$.
Wave-length in vacuum.	Intensity.			
4688.2	(1 broad)	21330	4358.3 } 4046.6 }	→1608 →3375
4642.9	(1)	21538	4358.3	1400
			(22938 cm. ⁻¹)	
4594.6	(2)	21765	4046.6	2940
			(24705 cm. ⁻¹)	
4531.4	(3)	22068	4358.3	870
4470.7	(1)	22368	"	570
4432.4	(0)	22561	"	377
4194.6	(1)	23840	4046.6	865
4141.9	(0)	24137	"	568

at $\Delta\nu=1400$. It is to be noticed that the prominent frequencies in acetic acid are slightly higher than those in acetamide. The shift at about $\Delta\nu=3360$ occurs

in both the aliphatic amides reported in this paper. This shift is known to be due to the NH_2 group.

The next amide examined belongs to the aromatic family. The Raman spectrum of benzamide is rich

TABLE III.
Benzamide (melts between 75° – 80° C.).
 $\text{C}_6\text{H}_5\text{CO}.\text{NH}_2$.

Modified lines.		Wave-number (cm^{-1}).	Exciting line.	Wave-number difference, $\Delta\nu$.
Wave-length in vacuum.	Intensity.			
4708.9	(0)	21236	4358.3 (22938 cm^{-1})	1702
4684.9	(3)	21345	$\rightarrow \left\{ \begin{array}{l} 4358.3 \\ 4046.6 \end{array} \right\}$	$\rightarrow 1593 \}$
4621.2	(2)	21639	$\rightarrow 4046.6$ (24705 cm^{-1})	$\rightarrow 3360 \}$ $\rightarrow 3066$
4597.6	(1)	21750	4358.3	1188
4559.1	(3)	21934	"	1004
4518.3	(0)	22132	"	806
4432.0	(1)	22563	"	375
4409.9	(0)	22676	"	262
4390.3	(0)	22777	"	161
4327.6	(1)	23107	4046.6	1598
4253.3	(0)	23511	"	1194
4218.7	(2)	23704	"	1001
4185.0	(0)	23895	"	810
<hr/>				
Benzoic acid, $\Delta\nu$ fused state.	161 (0), 375 (1), 806 (0), 1004 (3), 1188 (1), 1593 (3), 1702 (0), 3044 (1).			
<hr/>				
Benzamide, $\Delta\nu$ fused state.	167 (0), 798 (0), 1003 (3), 1172 (1), 1604 (3), 1687 (0), 3066 (2).			

in lines, and bears a close similarity to that of benzoic acid. The Raman spectrum of benzophenone in the fused state was also studied (see Table IV.).

The spectra of benzoic acid and benzamide are similar but differ from that of benzophenone, though all the three compounds contain the group $\text{C}_6\text{H}_5.\text{CO}$. The spectra of benzophenone is of the ketonic type, and compares well with that of acetophenone. In both these ketones the C_6H_5 group has reduced the frequency of the $\text{C}=\text{O}$ group near about $\Delta\nu=1700$.

The Raman spectra of urea in the fused state is very similar to that of the same compound studied in aqueous solution by Pal and Sengupta *.

TABLE IV.
Benzophenone (melting-point $42.2^{\circ}\text{C}.$).

Modified lines.		Wave-number (cm^{-1}).	Exciting line.	Wave-number difference, $\Delta\nu$.
Wave-length in vacuum.	Intensity.			
4697.5	(4)	21287	4358.3 (22938 cm^{-1})	1651
4685.5	(4)	21341	"	1597
4633.5	(1)	21582	"	1356
4616.1	(2 broad)	21663	4046.6 (24705 cm^{-1})	3042
4588.4	(4)	21794	4358.3	1144
4557.7	(4)	21940	"	998
4500.9	(1)	22217	"	721
4480.0	(1 broad)	22320	"	618
4402.7	(2)	22713	"	227

TABLE V.
Urea (melting-point $132^{\circ}\text{C}.$). $\begin{array}{c} \text{NH}_2 \\ \diagup \\ \text{CO} \\ \diagdown \\ \text{NH}_2 \end{array}$

Modified lines.		Wave-number (cm^{-1}).	Exciting line.	Wave-number difference, $\Delta\nu$.
Wave-length in vacuum.	Intensity.			
4688.2	(1)	21330	4046.6 (24705 cm^{-1})	3374
4656.2	(1)	21475	"	3230
4596.0	(0)	21758	4358.3 (22938 cm^{-1})	1180
4560.1	(2)	21929	"	1009
4219.9	(1)	23697	4046.6	1008

The Raman spectrum of acetanilide in the fused state contains a large number of lines. and the spectrum as a whole is very similar to that of aniline. This is to be expected, as annelides are derivatives of aniline.

* Ind. Journ. of Physics, v. p. 13 (1930).

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The shifts in the neighbourhood of $\Delta\nu=3300$ are strong in aniline, phenyl hydrazine, etc. These are absent in the spectrum of the annelide studied here. Absence of such higher shifts is to be noticed in the spectra of dimethyl and diethyl anilines, toluidine, xylidene, etc. The shift at 2939 in the annelide is due

TABLE VI.
Acetanilide (melting-point 113° C.).
 $\text{C}_6\text{H}_5\cdot\text{NH}\cdot\text{CO}\cdot\text{CH}_3$.

Modified lines.		Wave-number (cm^{-1}).	Exciting line.	Wave-number difference, $\Delta\nu$.
Wave-length in vacuum.	Intensity.			
4704.3	(0)	21257	4358.3 (22938 cm^{-1})	1681
4686.0	(4)	21340	"	1598
4623.6	(2)	21628	"	1310
4609.9	(1)	21692	4046.6 (24705 cm^{-1})	3013
4594.3	(1)	21766	"	2939
4559.5	(3)	21932	4358.3 (22938 cm^{-1})	1006
4527.5	(1)	22087	"	851
4511.2	(0)	22167	"	771
4488.9	(1)	22277	"	661
4429.7	(1)	22575	"	363
4407.8	(0)	22687	"	251
4326.9	(1)	23111	4046.6	1594
4274.2	(1)	23396	"	1309
4219.5	(2)	23699	"	1006
4192.8	(1)	23850	"	855
4159.4	(1)	24042	"	663
<hr/>				
Aniline, $\Delta\nu$..	233 (3), 385 (3), 762 (2), 813 (5 b), 993 (10), 1270 (4), 1600 (10).		
<hr/>				
Acetanilide, $\Delta\nu$ fused state.		251 (0), 363 (1), 771 (0), 851 (1), 1006 (3), 1310 (2), 1598 (4).		

to the aliphatic C-H group in the annelide, and the shift 3013 is due to the C-H group in the benzene ring.

Discussion of the Results.

The results obtained from the study of the Raman spectra of formamide and acetamide are in agreement with those given by Dadieu and Kohlrausch*.

* *Wien Ber.* cxxxviii, p. 635 (1929).

In the case of urea in the fused state the shifts reported in this paper are higher than those obtained from the study of the spectra of aqueous solutions of the same compound. This is probably due to the difference of temperature and the presence of the solvent media.

The shift in the region of $\Delta\nu=1700$ is present in the spectra of the three amides. It has a value 1675 in formamide, and 1702 in benzamide. Its lower value near about 1600 in acetamide is not understood. The shifts in the region $\Delta\nu=3300$ are due to the NH_2 group, and are conspicuous in the spectra of all compounds which contain the NH_2 group. It is not noticed in the spectrum of acetanilide, though this substance is a derivative of aniline and its spectrum is similar to that of aniline in other respects.

It would be interesting to study the Raman spectra of amides and imides of the polybasic acids; and experiments in this direction are in progress in this laboratory.

The authors wish to thank the authorities of the Nagpur University for a research grant to meet the expenses of this investigation.

Nagpur University,
December 1934.

XCV. *Note on the Stresses in some Rotating Circular Disks of varying Thickness.* By BIBHUTIBHUSAN SEN, Krishnagar College, Bengal*.

1. Introduction.

Average values of the stresses in thin rotating circular disks have been determined in the cases when the thickness $2h$ and the distance r from the centre are connected by the relations

$$h = h_0 r^{-k} \dagger,$$

$$\text{and} \quad h = h_0 (1 - br)^k \ddagger,$$

h_0 , b , and k being constants.

* Communicated by the Author.

† *Vide* Prescott's 'Applied Elasticity,' p. 343.

‡ "Stresses in a Rotating Circular Disc of small but varying Thickness," by T. Suhara and H. Yaskawa, Proceedings of the Imperial Academy of Japan, vol. v. p. 72.

The object of this paper is to find the average values of the stresses in thin rotating circular disks when

$$h = h_0 e^{-kr} \quad . \quad . \quad . \quad . \quad . \quad (1.1)$$

and
$$h = h_0 \left(1 - \frac{r^2}{b^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (1.2)$$

In the above relations h_0 , k , and b are constants.

Let P and Q be the mean radial and hoop stresses at a distance r from the centre and $2h$ the thickness at the point.

Then the equation* of motion assumes the form

$$r^2 \frac{d^2 y}{dr^2} + r \frac{dy}{dr} \left(1 - \frac{r}{h} \frac{dh}{dr}\right) + y \left(\sigma \frac{r}{h} \frac{dh}{dr} - 1\right) = -(3 + \sigma) \rho h r^3 \omega^2, \quad . \quad . \quad . \quad (1.3)$$

where $y = hrP$,

σ = Poisson's ratio,

ρ = density,

and ω = the angular velocity.

The stress Q is obtained from the relation

$$hQ = \frac{d(hrP)}{dr} + \rho h r^2 \omega^2. \quad . \quad . \quad . \quad (1.4)$$

2. Thickness varying as e^{-kr} .

Putting $h = h_0 e^{-kr}$ in the equation (1.3), and putting β for $(3 + \sigma) \rho h_0 \omega^2$, we get

$$\frac{d^2 y}{dr^2} + \frac{1}{r} \frac{dy}{dr} (1 + kr) - \frac{y}{r^2} (\sigma kr + 1) = -\beta e^{-kr} r. \quad (2.1)$$

If we write in this equation $y = e^{-kr} R$, we have

$$\frac{d^2 R}{dr^2} + \frac{1 - kr}{r} \frac{dR}{dr} - R \left[\frac{(1 + \sigma)k}{r} + \frac{1}{r^2} \right] = -\beta r. \quad (2.2)$$

As a particular integral of the equation we assume

$$R = C_1 r^2 + D_1 r. \quad . \quad . \quad . \quad (2.3)$$

* Prescott, *loc. cit.* p. 344.

We find that the equation is satisfied if

$$C_1 = \frac{\beta}{k(3+\sigma)} = \frac{\rho h_0 \omega^2}{k}, \quad . \quad . \quad . \quad (2.4)$$

$$D_1 = \frac{3C_1}{k(2+\sigma)} = \frac{3\rho h_0 \omega^2}{k^2 \alpha}, \quad . \quad . \quad . \quad (2.5)$$

where $\alpha = (2 + \sigma)$.

The complementary function is obtained from the equation

$$\frac{d^2 R}{dr^2} + \frac{1-kr}{r} \frac{dR}{dr} - R \left[\frac{(1+\sigma)k}{r} + \frac{1}{r^2} \right] = 0. \quad . \quad (2.6)$$

In this equation we put

$$R = xR_1, \quad \text{and} \quad kr = x.$$

Then the equation takes the simple form

$$x \frac{d^2 R_1}{dx^2} + \frac{dR_1}{dx} (3-x) - \alpha R_1 = 0. \quad . \quad . \quad (2.7)$$

The solution of this well-known equation is the confluent hypergeometric function $M(\alpha, 3, x)$, which stands for the convergent series

$$1 + \frac{\alpha}{1.3}x + \frac{\alpha(\alpha+1)}{1.2.3.4}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{1.2.3.3.4.5}x^3 + \dots \quad (2.8)$$

(The tables of this function have been given in the Reports of the Math. Tables Comm., British Association, 1927.)

We can then write

$$hrP = y = A_1 r e^{-kr} M(\alpha, 3, kr) + e^{-kr} (C_1 r^2 + D_1 r),$$

whence we get

$$P = \frac{A_1}{h_0} M(\alpha, 3, kr) + \frac{\rho r \omega^2}{k} + \frac{3\rho \omega^2}{k^2 \alpha}. \quad . \quad . \quad (2.9)$$

Since $P=0$ on the boundary $r=a$, we have

$$A_1 = - \frac{[3 + k\alpha] \rho h_0 \omega^2}{M(\alpha, 3, ka) k^2 \alpha}. \quad . \quad . \quad (2.10)$$

Knowing P we can determine Q from the relation

$$hQ = \frac{dy}{dr} + \rho h r^2 \omega^2.$$

Utilizing the results

$$\begin{aligned}\frac{d}{dx}[M(\alpha, \gamma, x)] &= \frac{\alpha}{\gamma} M(\alpha+1, \gamma+1, x) \\ &= \frac{\alpha-\gamma}{\gamma} [M(\alpha, \gamma+1, x)] + M(\alpha, \gamma, x),\end{aligned}$$

we find from the value of y given above

$$\begin{aligned}Q &= -\frac{[3+ka\alpha]\rho\omega^2}{3k^2\alpha M(\alpha, 3, \kappa\alpha)} [3M(\alpha, 3, kr) - (3-\alpha)krM(\alpha, 4, kr)] \\ &\quad + \frac{\rho\omega^2}{k^2\alpha} [3 + (2\alpha-3)kr]. \quad \dots \quad (2.11)\end{aligned}$$

3. Thickness varying as $\left(1 - \frac{r^2}{b^2}\right)$.

Putting $h = h_0 \left(1 - \frac{r^2}{b^2}\right)$ in the equation (1.3), we have

$$\begin{aligned}\left(1 - \frac{r^2}{b^2}\right) \frac{d^2y}{dr^2} + \frac{1}{r} \frac{dy}{dr} \left(1 + \frac{r^2}{b^2}\right) + \frac{y}{r^2} \left[(1-2\sigma) \frac{r^2}{b^2} - 1\right] \\ = -\beta r \left(1 - \frac{r^2}{b^2}\right)^2. \quad \dots \quad (3.1)\end{aligned}$$

If z is written for $\frac{r}{b}$ and λ for $(1-2\sigma)$ this equation becomes

$$(1-z^2) \frac{d^2y}{dz^2} + \frac{1}{z} \frac{dy}{dz} (1+z^2) + \frac{y}{z^2} [\lambda z^2 - 1] = -\beta b^3 z (1-z^2)^2. \quad (3.2)$$

The particular integral can be obtained by putting

$$y = B_2 z^5 + C_2 z^3 + D_2 z, \quad \dots \quad (3.3)$$

where B_2 , C_2 , and D_2 are constants.

Substituting this value of y in (3.2), we find that the equation is satisfied if

$$B_2 = D_2 = \frac{\beta b^3}{15-\lambda},$$

and

$$C_2 = -\frac{2\beta b^3}{15-\lambda}. \quad \dots \quad (3.4)$$

To find the complementary function, we are to solve the equation

$$(1-z^2)\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz}(1+z^2) + \frac{y}{z^2}(\lambda z^2-1)=0. \quad (3.5)$$

The solution of this equation in series can be obtained as

$$y=A_2\left\{z + \frac{[1.(-1)-\lambda]}{2.4}z^3 + \frac{[1.(-1)-\lambda]}{2.4} \cdot \frac{[3.1-\lambda]}{4.6}z^5 + \frac{[1.(-1)-\lambda]}{2.4} \cdot \frac{[3.1-\lambda]}{4.6} \cdot \frac{[5.3-\lambda]}{6.8}z^7 + \dots\right\}. \quad (3.6)$$

The series is convergent when

$$|z| \leq 1, \quad i.e., \quad \text{when } |r| \leq b.$$

Putting the series as $A_2F(z)$ or $A_2F\left(\frac{r}{b}\right)$, we get the complete value of

$$y=hrP=A_2F\left(\frac{r}{b}\right) + B_2\left(\frac{r}{b}\right)^5 + C_2\left(\frac{r}{b}\right)^3 + D_2\left(\frac{r}{b}\right),$$

whence

$$P = \frac{1}{h_0r\left(1-\frac{r^2}{b^2}\right)} \left[A_2F\left(\frac{r}{b}\right) + B_2\left(\frac{r}{b}\right)^5 + C_2\left(\frac{r}{b}\right)^3 + D_2\left(\frac{r}{b}\right) \right]. \quad (3.7)$$

We take the radius a of the disk $< b$, and assume $\frac{a}{b}=q$.

Since $P=0$ at the edge $r=a$, we have

$$\begin{aligned} A_2 &= - \frac{B_2q^5 + C_2q^3 + D_2q}{F(q)} \\ &= - \frac{B_2q(1-q^2)^2}{F(q)}, \quad \dots \quad (3.8) \end{aligned}$$

B_2 having the value given in (3.4).

On substituting the values of the constants A_2 , B_2 , C_2 , and D_2 in (3.7), P is completely determined, and then Q can be obtained from the formula given in (1.4).

XCVI. *Velocity of Waves in Oil under Pressure.* By
A. C. TALBOTT, *M.Sc., Lecturer in Engineering, Armstrong*
College, University of Durham *.

Introduction.

THE velocity of pressure waves in oil contained in pipes is of importance for Diesel engines running at high speeds, and it was primarily with this in view that the investigation was undertaken, on the suggestion of Prof. C. J. Hawkes. The work includes pressures up to 6000 lb. per sq. in., which covers the range usual in Diesel engines.

Theory.

The problem is essentially the same as that of measuring the velocity of sound, and the theoretical formulæ for gases have been used. These are based upon the fact that the displacements are small, which is always true in a liquid, even though there is a considerable rise in the pressure.

The effect of the viscous drag, due to the containing walls of a circular tube, can be expressed by the two formulæ first obtained by Helmholtz † :—

$$\text{Velocity, } v = \sqrt{\frac{K}{\rho}} \left(1 - \frac{1}{r} \sqrt{\frac{\mu}{2\rho\omega}} \right), \quad \dots (1)$$

$$\text{Attenuation factor} = \frac{1}{rv} \sqrt{\frac{\omega\mu}{2\rho}}, \quad \dots (2)$$

where

K = bulk modulus of fluid,

ρ = density of fluid,

$\omega = 2\pi \cdot$ frequency of waves,

μ = viscosity of fluid,

r = radius of containing tube.

In the proof of these formulæ the value of

$$\frac{J_1(y\sqrt{-i})}{J_0(y\sqrt{-i})},$$

where

$$y = r \sqrt{\frac{\rho\omega}{\mu}},$$

* Communicated by the Author.

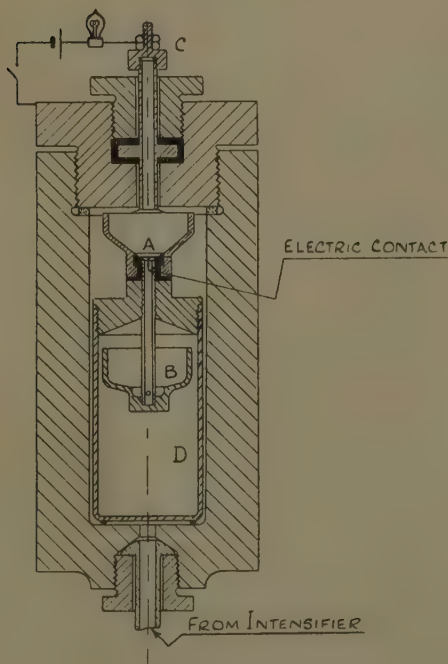
† Crandall, 'Vibrating Systems and Sound,' p. 234.

has been taken as equal to $-i$. This is approximately true when y is greater than 10. This condition applies in most of the cases to be considered.

It is seen from the formulæ that it is necessary to find the bulk modulus, the density, and the viscosity of the Diesel oil used. The experimental part, therefore, divides itself into three separate portions :—

- (1) Bulk modulus of oil.
- (2) Viscosity of oil under pressure.
- (3) Measurement of velocity of wave.

Fig. 1.



Bulk Modulus.

Apparatus.—This is shown diagrammatically in fig. 1. A copper tube was screwed to the cover of the thin steel vessel, D, and fitted with an insulated filling-cup, A,

at its upper end, and a copper pot, B, at its lower end. A piece of fine platinum wire was attached to the filling-cup, and passed a short way down the centre of the tube. The inner vessel was surrounded by a cylinder, strong enough to withstand the maximum pressure applied.

The liquid was placed in the inner vessel and mercury was poured into the tube and copper pot to form a mercurial seal. The space between the inner and outer vessels was filled with distilled water, and the pressure applied by a hydraulic intensifier. The pressure of the water was transmitted through the mercury to the liquid, so that the inner vessel contracted in volume as a solid

TABLE I.

Pressure, lb. per sq. in. (above atmos. press.).	Volumes in terms of original volume.					
	32° F.	60° F.	90° F.	120° F.	150° F.	180° F.
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1000	0.99625	0.99660	0.99685	0.99697	0.99697	0.99681
2000	0.99281	0.99342	0.99387	0.99408	0.99409	0.99391
3000	0.98944	0.99033	0.99097	0.99124	0.99127	0.99108
4000	0.98615	0.98731	0.98813	0.98846	0.98851	0.98830
5000	0.98293	0.98437	0.98530	0.98573	0.98580	0.98557
6000	0.97988	0.98148	0.98257	0.98305	0.98312	0.98287

under hydrostatic pressure. The apparatus was connected up electrically as shown in the diagram, and the electric circuit was completed when the mercury was in contact with the platinum wire. The whole apparatus was surrounded by a water-bath.

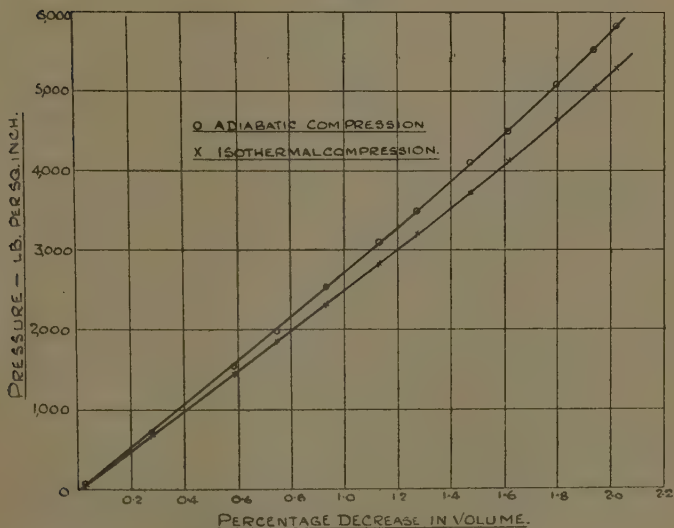
The cap, C, was removed and mercury added or taken away until the electric contact between the platinum wire and the mercury was broken at atmospheric pressure. A weighed quantity, and, therefore, a known volume, of mercury was added to the filling-cup, and the pressure then necessary to break the contact was found.

The pressure was measured by an optical gauge, which will be described later, and a lamp indicated when contact was broken. A telescope and scale were used

for the pressure gauge, and the lamp was placed in the field of view of the telescope.

Compressibility of Water.—These experiments were made, in order to check the accuracy of the apparatus, before commencing the experiments on oil. There was very little difference between the adiabatic and the isothermal compressions in the temperature range considered. The results are given in Table I.

Fig. 2.



Compression of Diesel oil at 32.3° F.

These figures appear to agree with those of Amagat * at a temperature of about 100° F., and are slightly lower than those obtained by Tammann and Jellinghaus † in the temperature range they considered.

Compressibility of Diesel Oil.—The compressibility required was that for adiabatic compression, but several tests were made for isothermal compressions. Fig. 2 shows the decrease in volume for 100 c.c. for

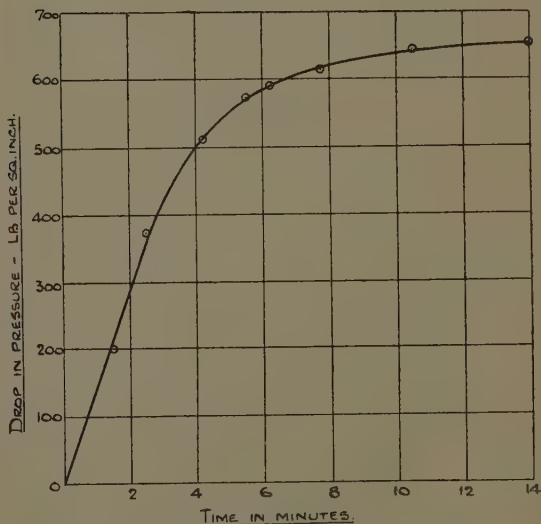
* *Ann. Chim. Phys.* xxix. pp. 545, 505 (1893).

† *Z. anorg. allgem. Chem.* clxxiv. p. 225 (1928).

adiabatic and isothermal compressions at an initial temperature of 32.3°F .

A graph, showing the drop in pressure plotted against time, after an adiabatic compression to 5900 lb. per sq. in., is shown in fig. 3. The temperature before compression and after cooling was 50°F . The maximum pressures in the tests were applied in ten seconds, so that there

Fig. 3.



Cooling curve showing compression tests on oil.

appears to be a maximum error of 20 lb. per sq. in. in the adiabatic pressures recorded, for which no allowance has been made.

The rise in temperature, due to compression, was obtained from the thermodynamic relationship :—

$$\text{Head added} = K_P dT - \int \left(\frac{\partial V}{\partial T} \right)_P dP^*.$$

* Ewing, 'Thermodynamics for Engineers,' p. 274.

For adiabatic compression,

$$0 = K_P dT - \frac{T}{J} \left(\frac{\partial V}{\partial T} \right)_P dP.$$

Let the coefficient of expansion

$$= \alpha,$$

$$= \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_P dP.$$

Then

$$K_P dT = \frac{TV\alpha}{J} dP,$$

from which

$$\frac{dT}{dP} = \frac{TV\alpha}{JK_P}.$$

The value of the specific heat at constant pressure was obtained for an average oil from the equation *

$$K_P = \frac{0.415}{\sqrt{\rho}} + 0.0005 (t - 59) \text{ per } ^\circ \text{F.},$$

where

ρ = specific gravity of oil at 59°F. ,

t = temperature in $^\circ \text{F.}$

The value of the coefficient of expansion of the oil was determined experimentally at atmospheric pressure. The approximation for the mean value at constant pressure for each compression was obtained by the method adopted by Parsons and Cook †.

The experimental results have been corrected for decrease in volume of the containing pot, the compressibility of the mercury, and for the rise in temperature due to compression. The figures obtained for the oil are shown in Table II.

The figures shown in this table have been used in calculating the bulk modulus from the equation

$$K = \frac{dP}{dv} \cdot V,$$

* 'International Critical Tables,' ii. p. 151.

† Proc. Roy. Soc. A, lxxxiv. (1911).

where

 K = bulk modulus, P = rise in pressure, v = decrease in volume, V = original volume.

The value of $\frac{dP}{dv}$ was taken as the slope of the compressibility curve at each point considered. Fig. 4

TABLE II.

Pressure, lb. per sq. in. (above atmos. press.)	Volumes in terms of original volume.					
	42° F.	70° F.	110° F.	150° F.	190° F.	210° F.
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
500	0.99798	0.99782	0.99754	0.99716	0.99662	0.99628
1000	0.99604	0.99572	0.99516	0.99444	0.99344	0.99282
1500	0.99416	0.99370	0.99288	0.99184	0.99042	0.98958
2000	0.99234	0.99174	0.99068	0.98936	0.98758	0.98656
2500	0.99058	0.98982	0.98854	0.98696	0.98486	0.98366
3000	0.98886	0.98796	0.98646	0.98464	0.98224	0.98088
3500	0.98718	0.98614	0.98442	0.98238	0.97970	0.97820
4000	0.98554	0.98436	0.98244	0.98018	0.97724	0.97562
4500	0.98394	0.98262	0.98050	0.97804	0.97484	0.97312
5000	0.98236	0.98092	0.97862	0.97596	0.97252	0.97068
5500	0.98080	0.97926	0.97678	0.97392	0.97026	0.96830
6000	0.97928	0.97764	0.97498	0.97192	0.96806	0.96598

shows the values of the bulk modulus for lines of constant pressure.

The specific gravity of the Diesel oil at atmospheric pressure was plotted against the temperature, and found to be a straight line. The values were 0.8586 and 0.8004 at 50° F. and 200° F., respectively. The specific gravities of the oil at higher pressures were calculated from these values and the compressibility figures. The results are shown plotted in fig. 5 for lines of constant pressure.

Fig. 4.

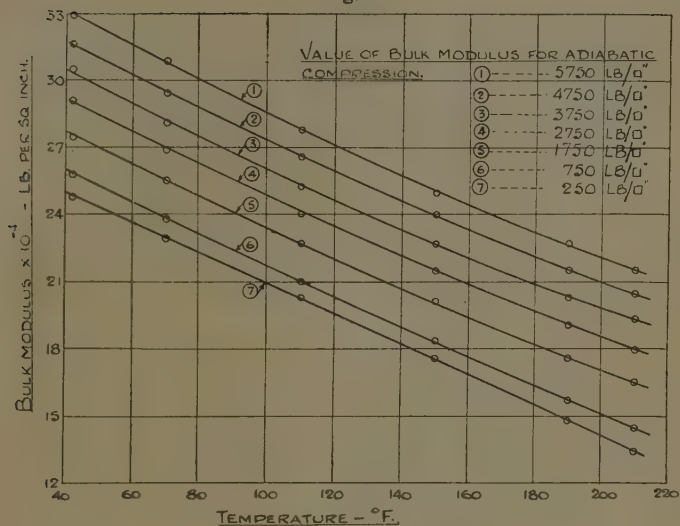
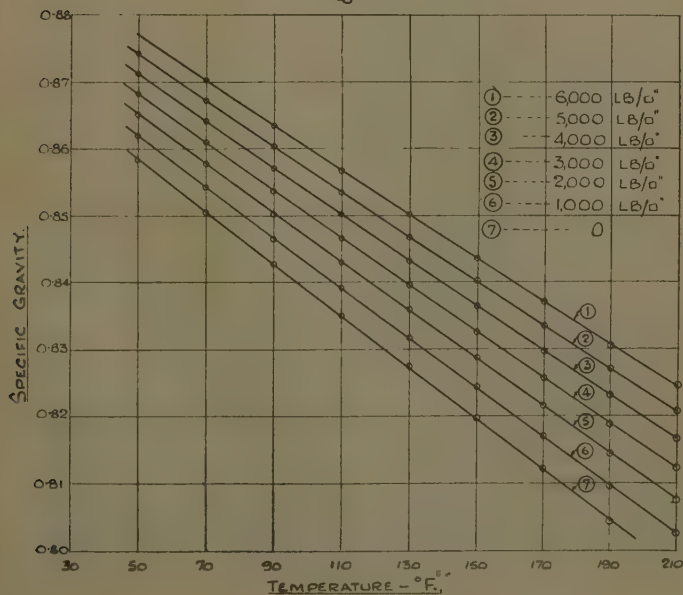


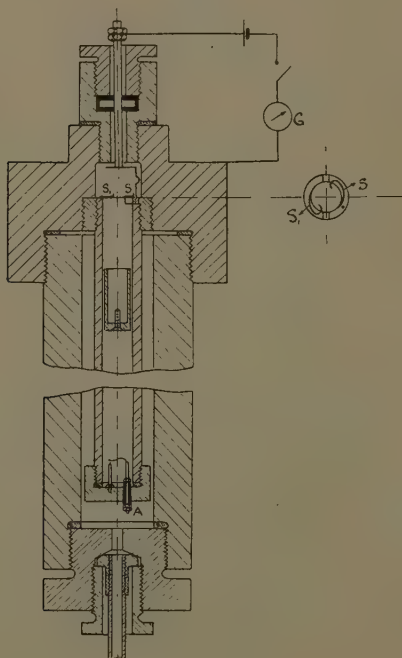
Fig. 5.



Viscosity.

The method adopted was to allow a piston to drop down a vertical tube, first in distilled water at atmospheric pressure, and then in the oil at the same temperature. The viscosities were taken as being proportional to the times taken for the piston to drop through the same vertical distance.

Fig. 6.



Apparatus.—This is shown in fig. 6. It consisted of a cylindrical tube, 0.75 in. bore and 2 ft. in length, fitted with a light alloy piston. Two weak springs, S and S_1 , of the form shown in the plan, were fixed to the top of the tube, and one was insulated. Two vertical legs, one of which was insulated, projected upwards from the bottom. The tube was surrounded by oil,

contained in a strong outer vessel, so that the tube had a minimum strain when the pressure was applied. The two insulated parts, A and S, were connected together, and to the external electric circuit shown in the diagram.

The electric circuit was closed when the piston was in contact with the springs at the top, or with the two legs at the bottom. The times taken for the piston to fall through the fixed vertical distance were obtained by a stop-watch.

Values of the viscosity of distilled water at atmospheric pressure were obtained from 'International Critical Tables,' v. p. 10, and the viscosity of Diesel oil was

TABLE III.

Pressure, lb. per sq. in. (above atmos. press.).	Viscosity of Diesel oil in millipoises.						
	60° F.	85° F.	110° F.	135° F.	160° F.	185° F.	210° F.
0	54.85	34.19	24.61	18.51	14.22	11.17	9.00
1000	61.25	37.98	27.13	20.28	15.52	12.17	9.78
2000	68.25	42.12	29.82	22.19	16.92	13.24	10.63
3000	75.92	46.66	32.69	24.24	18.42	14.39	11.54
4000	84.37	51.66	35.77	26.42	20.01	15.62	12.51
5000	93.79	57.20	39.13	28.73	21.70	16.93	13.54
6000	104.54	63.41	42.85	31.18	23.49	18.33	14.63

determined at different pressures throughout the temperature range. The results agreed perfectly with those obtained at atmospheric pressure with a normal type of absolute viscometer.

The oil under pressure increased the specific gravities of the oil and the piston, and decreased the thickness of the oil-film between the piston and the tube. These very small corrections have been made, and the results are shown in Table III.

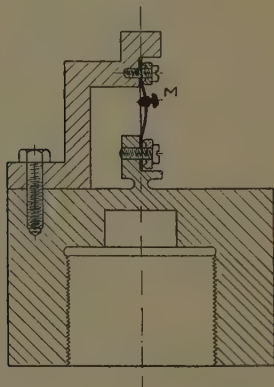
Measurement of Wave-Velocity.

Method.—Oil was contained in a pipe, one end of which was closed by the pressure-recording gauge, and the other end had a non-return spring-loaded valve. The

pressure-wave was produced by forcing a small quantity of oil through the valve. The wave was reflected at the ends and passed backwards and forwards along the pipe until it was damped out.

Apparatus.—The pressure-recording gauge, shown in fig. 7, depended upon the deformation of a diaphragm, but, as the maximum deflexion was only 0.0003 in., it responded quickly, and the slight increase in volume caused very little change in the pressure inside the pipe. A piece of thin steel had one end clamped to the diaphragm,

Fig. 7.



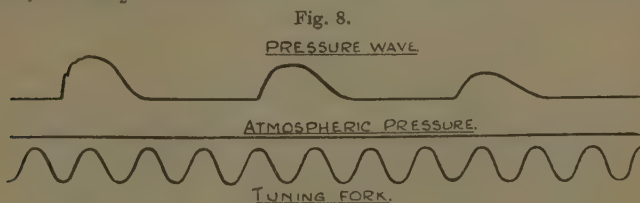
and the other end to a stationary part of the gauge. The piece of steel acted as a strut, fixed at the ends, and a mirror was attached at the point which moved through the maximum angle when the strut was compressed. The pressure-gauge was calibrated before each set of experiments, and the graph in each case was almost a straight line.

The time-scale was obtained by a plate falling under gravity. The plate was released by an electro-magnet, and, after falling $3\frac{1}{2}$ ft., had to pass through a hole, each side of which was 0.05 in. larger than the corresponding side of the plate. This ensured that the plate fell in the same plane for each experiment. The plate was checked for time against a tuning fork of known frequency, and

it was found to fall with an acceleration of 32.092 ft./sec.² This figure was used for calculating the times of fall, and, in addition, the times were checked by means of a tuning fork.

An incident beam of light was reflected through three mirrors, so that three point-images were formed on the plane through which the plate passed. The first mirror was attached to the pressure-gauge, the second to a fixed part of the apparatus to form the zero pressure line, and the third was fixed to the tuning fork. A sensitive film was fastened to the plate for each experiment, and the spots of light produced a record as the film dropped.

Experiments were made throughout the temperature-range from 65° F. to 205° F. on four pipes of diameters 0.245, 0.186, 0.131, and 0.076 in., each 40 ft. in length, and on the largest diameter pipe for lengths of 10, 17½, 25, and 32½ ft.



Results.—A typical film for a pipe of diameter 0.245 in. and 40 ft. in length is shown in fig. 8. The horizontal portion of the curve was less for the shorter pipes, and did not appear in pipes 10 ft. in length. The wave-form was similar to that obtained in the pipe on an engine using an oil-operated sprayer.

There was very little difference between the velocities of the peak and of the base, and the mean velocity was taken. The mean pressure under each curve was used in plotting the results.

The velocity of a pressure-wave was calculated for oil in a tube of infinite radius, i. e., $v = \sqrt{\frac{K}{\rho}}$, and these graphs are shown in figs. 9 and 10. The experimental value of the velocity was reduced slightly on account of

- (a) the damping due to the viscous drag,
- (b) the expansion of the containing pipe.

Fig. 9.

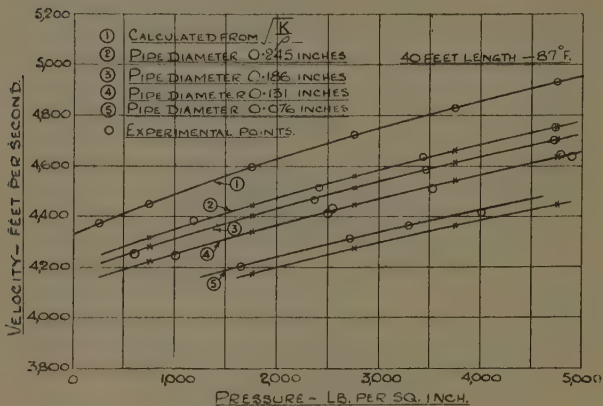
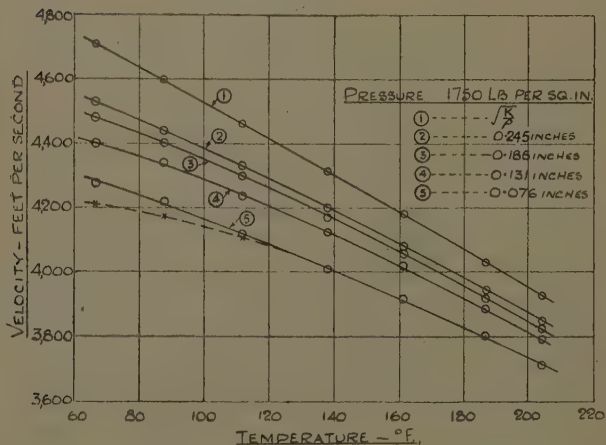


Fig. 10.



The figures were obtained for the damping according to equation (1), the frequency being taken as the velocity divided by twice the length of the pipe.

The expansion of the pipe modifies the bulk modulus

of the oil. The correction for a thin metal pipe is usually taken as

$$\frac{1}{K'} = \left(\frac{1}{K} + \frac{1D}{Ed} \right)^*, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where

K = true bulk modulus of the oil,

K' = modified bulk modulus of the oil,

d = inner diameter of the pipe,

D = outer diameter of the pipe,

E = Young's modulus for the metal.

The correction for a thick pipe can be worked out in the same way, and is

$$\frac{1}{K'} = \frac{2}{E} \left(\frac{D^2 + d^2}{D^2 - d^2} + \frac{1}{m} \right) + \frac{1}{K}, \quad . \quad . \quad . \quad (4)$$

where

m = Poisson's ratio.

These corrections are based on the assumption that the pipe expands as a cylinder under a uniform internal static pressure acting throughout its whole length. A pressure-wave is only momentary, and is localised, so that the effect must be less than that given by these equations. This has been shown to be true experimentally, and work is in progress on the subject which will be published shortly.

The experiments were made in steel pipes which had the external diameter twice the internal diameter. Equation (4) produced a reduction of about $1\frac{1}{2}$ per cent. in the velocity of the waves.

Half the values obtained from equation (4) have been used as corrections for the expansion of the pipe, and the figures obtained from equation (1) are shown by the crosses in figs. 9 and 10. It is seen in these graphs that there is complete agreement between the experimental and the calculated values, except for the smallest pipe, at the lower temperatures. The region in which the difference occurs is outside the limiting value mentioned on p. 1127.

* *Stress, 'Kompressorlose Dieselmotoren,'* p. 212.

The damping of the pressure-wave cannot be expected to be quite the same as that given in equation (2), as the wave-form is very different from that assumed in the derivation of the formula. The damping is a change of energy, and, therefore, is represented by an area on the pressure-space diagram. The areas between the wave and its mean base-line have been taken in calculating the attenuation factor.

The only satisfactory curves for measuring were those taken on the largest diameter pipe, as the other curves changed in form considerably, and it was uncertain which areas should be taken. The attenuation factor for the largest diameter pipe was found to agree with equation (1) for the pipe 40 ft. in length, but increased slightly with increase in frequency, and was about 15 per cent. greater with the pipe 17½ ft. in length. This may be due to different wave-forms, but it is most probably due to the greater number of reversals. Energy is lost on each reversal, and the damping effect due to this can be shown as follows.

Let the kinetic energy of the fluid = E , then loss of energy at each reversal = AE , where A is a constant.

Let the frequency of reversals = f , then

$$\frac{dE}{dt} = AEf,$$

from which

$$E = E_1 e^{fAt} \dots \dots \dots (5)$$

This increases the attenuation factor given by equation (2) by a term proportional to the frequency of reversal, but does not affect the velocity of the pressure-wave.

The pressure-wave changed slightly in shape in successive waves, tending to spread out and make the pressure more uniform. The pressure-amplitude, therefore, changed at a greater rate than given by equation (2), but was of the same form. The reduction in the pressure-amplitude was about 20 per cent. greater than that given by equation (2) at 66° F., and 70 per cent. greater at 204° F. This greater reduction in pressure-amplitude was noted by Simmons and Johansen* in their experiments on very low-frequency air-waves, and this spreading of the pressure-wave may account for their observations.

* Phil. Mag. 1. p. 553 (1925).

Conclusions.

The experiments indicate that equations (1) and (2) apply in most cases for airless-injection oil engines. There is a rise in velocity with increase in pressure, and a decrease in velocity with increase in temperature. The effect of viscosity reduces the velocity of the pressure-wave by about ten per cent. in the extreme cases, and, although the attenuation factor is approximately the same as given by equation (2), the pressure-amplitude is damped at a greater rate.

The temperature conductivity dealt with by Kirchhoff produces very little effect for oil. It can be shown that by substituting for the kinematic viscosity the expression *

$$\nu = \sqrt{\frac{\mu}{\rho}} + \left(\sqrt{\frac{K}{K_1}} - \sqrt{\frac{K}{K}} \right) \sqrt{\nu}, \quad (6)$$

where

K = bulk modulus for adiabatic compression,

K_1 = bulk modulus for isothermal compression,

ν = temperature conductivity of fluid,

in equations (1) and (2), allowance can be made for the temperature conductivity.

Fig. 2 indicates that there is only a small difference between the adiabatic and isothermal compressibility of Diesel oil. Very little information is available on temperature conductivity, but for most oils it is between one and two per cent. of the value of the kinematic viscosity of Diesel oil at ordinary temperatures and pressures. Both factors in the second term of equation (6) are small, so that the effect of the temperature conductivity is almost negligible, and the simpler equations of Helmholtz can be used.

I wish to thank the Council of Armstrong College for grants received from the College Research Fund. My thanks are also due to Prof. C. J. Hawkes for many helpful suggestions and the interest he has taken in the work; to Mr. R. Hiscock for reproducing the diagrams; and to Mr. C. B. W. Willson, B.Sc., for his valuable assistance in measuring many of the films.

* Rayleigh, 'Theory of Sound,' ii. p. 319.

XCVII. *On some Trigonometric Series.* By T. M. MACROBERT, *Professor of Mathematics, University of Glasgow* *.

MOST of the following results are to be found in Bromwich's 'Infinite Series' (second edition), §§ 124, 125 and pages 392, 393. The method of proof given here, involving the use of Dirichlet integrals, is shorter and simpler than that employed by Bromwich. It may be noted that Bromwich made use of the Dirichlet integral method in the section (126) immediately following.

The identity

$$\sum_{n=-m}^m \cos(n+\alpha)\theta = \cos(\alpha\theta) \frac{\sin(m+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta},$$

where m is a positive integer and α is any number, leads on integration to the equation

$$\sum_{n=-m}^m \frac{\sin(n+\alpha)\theta}{n+\alpha} = \int_0^\theta \cos(\alpha\theta) \frac{\theta}{\sin\frac{1}{2}\theta} \frac{\sin(m+\frac{1}{2})\theta}{\theta} d\theta.$$

If $0 < \theta < 2\pi$, the integral on the right is a Dirichlet integral; thus, when $m \rightarrow \infty$, the equation becomes

$$\sum_{n=-\infty}^{\infty} \frac{\sin(n+\alpha)\theta}{n+\alpha} = \pi, \quad 0 < \theta < 2\pi. \quad (1)$$

From this, by making $\alpha \rightarrow 0$, the well-known result

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = \frac{\pi - \theta}{2}, \quad 0 < \theta < 2\pi, \quad (2)$$

can be deduced.

Again, the identity

$$\sum_{n=-m}^m \sin(n+\alpha)\theta = \sin(\alpha\theta) \frac{\sin(m+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}$$

leads to the equation

$$\sum_{n=-m}^m \frac{1 - \cos(n+\alpha)\theta}{n+\alpha} = \int_0^\theta \sin(\alpha\theta) \frac{\theta}{\sin\frac{1}{2}\theta} \frac{\sin(m+\frac{1}{2})\theta}{\theta} d\theta.$$

If the L.H.S. is written in the form

$$\left\{ \frac{1}{\alpha} + \sum_{n=1}^m \frac{2\alpha}{\alpha^2 - n^2} \right\} - \sum_{n=-m}^m \frac{\cos(n+\alpha)\theta}{n+\alpha}.$$

* Communicated by the Author.

where α is not integral and θ is not a multiple of 2π ; and if m tends to ∞ , it is found that

$$\sum_{n=-\infty}^{\infty} \frac{\cos(n+\alpha)\theta}{n+\alpha} = \pi \cot(\alpha\pi), \quad \dots \quad (3)$$

where $0 < \theta < 2\pi$ or $-2\pi < \theta < 0$, and α is not integral.

From (1) and (3) it follows that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha} e^{i(n+\alpha)\theta} = \frac{\pi}{\sin(\alpha\pi)} e^{i\alpha\pi}, \quad \dots \quad (4)$$

where $0 < \theta < 2\pi$, and α is not integral.

If now $2p\pi < \theta < 2(p+1)\pi$, where p is any integer, (4) becomes

$$\sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha} e^{i(n+\alpha)\theta} = \frac{\pi}{\sin(\alpha\pi)} e^{i(2p+1)\alpha\pi}; \quad \dots \quad (5)$$

and, further,

$$\sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha} e^{i(n+\beta)\theta} = \frac{\pi}{\sin(\alpha\pi)} e^{i\{(2p+1)\alpha\pi + (\beta-\alpha)\theta\}}. \quad \dots \quad (6)$$

From (6) it follows that

$$\sum_{n=-\infty}^{\infty} \frac{\cos(n+\beta)\theta}{n+\alpha} = \frac{\pi}{\sin(\alpha\pi)} \cos\{(2p+1)\alpha\pi + (\beta-\alpha)\theta\}, \quad (7)$$

and

$$\sum_{n=-\infty}^{\infty} \frac{\sin(n+\beta)\theta}{n+\alpha} = \frac{\pi}{\sin(\alpha\pi)} \sin\{(2p+1)\alpha\pi + (\beta-\alpha)\theta\}. \quad (8)$$

Next, in (6), with $p=0$, put $\beta=0$ and $\alpha=ix$; then

$$\sum_{n=-\infty}^{\infty} \frac{n-ix}{n^2+x^2} e^{in\theta} = \frac{\pi}{i \sinh(\pi x)} e^{-(\pi-\theta)x}.$$

Now multiply by i , and get

$$\frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x \cos n\theta - 2n \sin n\theta}{n^2+x^2} = \frac{\pi}{\sinh(\pi x)} e^{-(\pi-\theta)x}.$$

On adding and subtracting the corresponding equation with $-x$ in place of x , we find that

$$\sum_{n=1}^{\infty} \frac{n \sin n\theta}{n^2+x^2} = \frac{\pi \sinh(\pi-\theta)x}{2 \sinh(\pi x)}, \quad \dots \quad (9)$$

and

$$\frac{1}{2x} + \sum_{n=1}^{\infty} \frac{x \cos n\theta}{n^2+x^2} = \frac{\pi \cosh(\pi-\theta)x}{2 \sinh(\pi x)} \dots \quad (10)$$

Again, on integrating (4), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{i(n+\alpha)^2} \{e^{i(n+\alpha)\theta} - 1\} = \frac{\pi\theta}{\sin(\alpha\pi)} e^{i\alpha\pi},$$

or

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} e^{i(n+\alpha)\theta} = \frac{\pi^2}{\sin^2(\alpha\pi)} + \frac{i\pi\theta}{\sin(\alpha\pi)} e^{i\alpha\pi}.$$

Hence, if $0 \leq \theta \leq 2\pi$,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} e^{i(n+\beta)\theta} &= \frac{\pi}{\sin(\alpha\pi)} \{ \pi \cot(\alpha\pi) - i(\pi - \theta) \} \\ &\quad \times e^{i\alpha\pi + i(\beta - \alpha)\theta}; \quad \dots \quad (11) \end{aligned}$$

and, consequently,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{\cos(n+\beta)\theta}{(n+\alpha)^2} &= \frac{\pi}{\sin(\alpha\pi)} \left[\pi \cot(\alpha\pi) \cos\{\alpha\pi + (\beta - \alpha)\theta\} \right. \\ &\quad \left. + (\pi - \theta) \sin\{\alpha\pi + (\beta - \alpha)\theta\} \right], \quad \dots \quad (12) \end{aligned}$$

and

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{\sin(n+\beta)\theta}{(n+\alpha)^2} &= \frac{\pi}{\sin(\alpha\pi)} \left[\pi \cot(\alpha\pi) \sin\{\alpha\pi + (\beta - \alpha)\theta\} \right. \\ &\quad \left. - (\pi - \theta) \cos\{\alpha\pi + (\beta - \alpha)\theta\} \right]. \quad \dots \quad (13) \end{aligned}$$

In formulæ (5) to (8) and (11) to (13) α must not be an integer.

In the next place, divide (4) by $e^{i\alpha\theta}$, and subtract it from the same formula with β in place of α ; then, if $0 \leq \theta \leq 2\pi$,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{(n+\alpha)(n+\beta)} &= \frac{\pi}{(\alpha - \beta) \sin(\alpha\pi) \sin(\beta\pi)} \\ &\quad \times \{e^{i\beta(\pi - \theta)} \sin(\alpha\pi) - e^{i\alpha(\pi - \theta)} \sin(\beta\pi)\}. \quad \dots \quad (14) \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{\cos n\theta}{(n+\alpha)(n+\beta)} &= \frac{\pi}{(\alpha - \beta) \sin(\alpha\pi) \sin(\beta\pi)} \\ &\quad \times \{ \cos(\pi - \theta) \beta \sin(\alpha\pi) - \cos(\pi - \theta) \alpha \sin(\beta\pi) \}, \quad (15) \end{aligned}$$

and

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{\sin n\theta}{(n+\alpha)(n+\beta)} &= \frac{\pi}{(\alpha - \beta) \sin(\alpha\pi) \sin(\beta\pi)} \\ &\quad \times \{ \sin(\pi - \theta) \beta \sin(\alpha\pi) - \sin(\pi - \theta) \alpha \sin(\beta\pi) \}. \quad (16) \end{aligned}$$

Now in (15) and (16) put $\alpha = x + iy$, $\beta = x - iy$; then

$$\sum_{n=-\infty}^{\infty} \frac{y \cos n\theta}{(n+x)^2 + y^2} = \frac{2\pi}{\cosh(2\pi y) - \cos(2\pi x)} \\ \times \left\{ \begin{aligned} &\cos(\pi - \theta)x \cosh(\pi - \theta)y \cos(\pi x) \sinh(\pi y) \\ &+ \sin(\pi - \theta)x \sinh(\pi - \theta)y \sin(\pi x) \cosh(\pi y) \end{aligned} \right\},$$

and

$$\sum_{n=-\infty}^{\infty} \frac{y \sin n\theta}{(n+x)^2 + y^2} = \frac{2\pi}{\cosh(2\pi y) - \cos(2\pi x)} \\ \times \left\{ \begin{aligned} &\sin(\pi - \theta)x \cosh(\pi - \theta)y \cos(\pi x) \sinh(\pi y) \\ &- \cos(\pi - \theta)x \sinh(\pi - \theta)y \sin(\pi x) \cosh(\pi y) \end{aligned} \right\}.$$

Hence

$$\sum_{n=-\infty}^{\infty} \frac{y \cos n\theta}{(n+x)^2 + y^2} = \frac{\pi}{\cosh(2\pi y) - \cos(2\pi x)} \\ \times \{ \cos(\theta x) \sinh(2\pi - \theta)y + \cos(2\pi - \theta)x \sinh(\theta y) \}, \quad (17)$$

and

$$\sum_{n=-\infty}^{\infty} \frac{y \sin n\theta}{(n+x)^2 + y^2} = \frac{\pi}{\cosh(2\pi y) - \cos(2\pi x)} \\ \{ \sin(2\pi - \theta)x \sinh(\theta y) - \sin(\theta x) \sinh(2\pi - \theta)y \}. \quad (18)$$

Again, from the identity

$$\sum_{n=0}^m \cos(x+n)\theta = \cos(x + \frac{1}{2}m)\theta \frac{\sin \frac{1}{2}(m+1)\theta}{\sin \frac{1}{2}\theta},$$

it follows by integration that

$$\sum_{n=0}^m \frac{\sin(x+n)\pi}{x+n} = \sum_{n=0}^m \frac{\sin(x+n)\theta}{x+n} \\ = \frac{1}{2} \int_{\theta}^{\pi} \cos(x - \frac{1}{2}\theta) \frac{\sin(m+1)\theta}{\sin \frac{1}{2}\theta} d\theta \\ - \frac{1}{2} \int_{\theta}^{\pi} \frac{\sin(x - \frac{1}{2}\theta)}{\sin \frac{1}{2}\theta} \{1 - \cos(m+1)\theta\} d\theta.$$

Hence, when $m \rightarrow \infty$, if $0 < \theta < 2\pi$,

$$\sum_{n=0}^{\infty} \frac{\sin(x+n)\theta}{x+n} = \frac{1}{2} \int_{\theta}^{\pi} \frac{\sin(x - \frac{1}{2}\theta)}{\sin \frac{1}{2}\theta} d\theta + \sin(\pi x) \sum_{n=0}^{\infty} \frac{(-1)^n}{x+n}.$$

(19)

Similarly, in the same range,

$$\sum_{n=0}^{\infty} \frac{\cos(x+n)\theta}{x+n} = \frac{1}{2} \int_{\theta}^{\pi} \frac{\cos(x - \frac{1}{2}\theta)}{\sin \frac{1}{2}\theta} d\theta + \cos(\pi x) \sum_{n=0}^{\infty} \frac{(-1)^n}{x+n}.$$

(20)

These results, (19) and (20), are due to Professor G. H. Hardy, as is also the limit theorem

$$\lim_{\theta \rightarrow 0} \sum_{n=0}^{\infty} \frac{\sin(x+n)\theta}{x+n} = \frac{1}{2}\pi, \quad 0 < \theta < 2\pi, \quad \dots \quad (21)$$

which can be proved as follows.

As in the proof of (19)

$$\begin{aligned} \sum_{n=0}^m \frac{\sin(x+n)\theta}{x+n} &= \frac{1}{2} \int_0^\theta \frac{\cos(x-\frac{1}{2})\theta}{\sin \frac{1}{2}\theta} \frac{\sin(m+1)\theta}{\sin \frac{1}{2}\theta} d\theta \\ &\quad - \frac{1}{2} \int_0^\theta \frac{\sin(x-\frac{1}{2})\theta}{\sin \frac{1}{2}\theta} \{1 - \cos(m+1)\theta\} d\theta; \end{aligned}$$

and, therefore,

$$\sum_{n=0}^{\infty} \frac{\sin(x+n)\theta}{x+n} = \frac{1}{2}\pi - \frac{1}{2} \int_0^\theta \frac{\sin(x-\frac{1}{2})\theta}{\sin \frac{1}{2}\theta} d\theta, \quad 0 < \theta < 2\pi. \quad (22)$$

From this (21) follows.

Formula (1) can also be derived by means of the contour integral

$$\frac{1}{2\pi i} \int \frac{\sin z \theta dz}{z \sin \pi(z-\alpha) \sin \pi(z-\beta)},$$

taken round a circle with the origin as centre and not passing through a pole of the integrand. If $-2\pi < \theta < 2\pi$ the integral tends to zero when the radius tends to infinity; hence

$$\sum_{n=-\infty}^{\infty} \frac{\sin(n+\alpha)\theta}{n+\alpha} = \sum_{n=-\infty}^{\infty} \frac{\sin(n+\beta)\theta}{n+\beta}, \quad -2\pi < \theta < 2\pi. \quad (23)$$

From (23), by making $\beta \rightarrow 0$ and employing the known formula (2), (1) can be obtained.

Similarly, from the integral

$$\frac{1}{2\pi i} \int \frac{\cos z \theta dz}{z \sin \pi(z-\alpha) \sin \pi(z-\beta)},$$

it can be deduced that, if $-2\pi < \theta < 2\pi$,

$$\sum_{n=-\infty}^{\infty} \left\{ \frac{\cos(n+\alpha)\theta}{n+\alpha} - \frac{\cos(n+\beta)\theta}{n+\beta} \right\} = \frac{\pi \sin(\beta-\alpha)\pi}{\sin(\alpha\pi) \sin(\beta\pi)}; \quad \dots \quad (24)$$

and from this, by putting $\beta = \frac{1}{2}$, (3) is obtained.

XCVIII. *On the Destructive Mechanical Effects of the Gas-bubbles liberated by the Passage of Intense Sound through a Liquid.* By F. D. SMITH, D.Sc., M.I.E.E.*

SUMMARY.

THE spherical bubbles of gas liberated by the passage of intense sound through liquids pulsate radially under the influence of the alternating sound pressure. The pulsating bubble has potential energy arising from the elasticity of the gas and kinetic energy arising from the movement of liquid in its vicinity. It therefore resonates at the frequency at which the mean potential and kinetic energies are equal. This frequency f is given by the formula

$$f = \frac{1}{2\pi a} \sqrt{\frac{3\kappa(p_0 + 2T/a)}{\rho}},$$

where a is the radius of the bubble, p_0 is the hydrostatic pressure on it, ρ is the density of the liquid, T is the surface tension of the liquid gas interface, and κ is the ratio of the specific heats of the gas. The following table gives the radius of the resonant bubble in water at various frequencies:—

Frequency .. f	$10^3 \sim$	$10^4 \sim$	$10^5 \sim$	$10^6 \sim$
Radius..... a	3.3 mm.	0.33 mm.	0.033 mm.	0.0038 mm.

The ratio of the mass reactance of the resonant bubble to the radiation resistance is 70; this corresponds to a high degree of resonance.

The resonant bubble is characterized by violent activity and shoots rapidly to and fro in the liquid, appearing to the eye as a luminous streak on account of persistence of vision; larger and smaller bubbles are seen to be relatively quiescent.

It is suggested that the well-known destructive mechanical effects of intense sound on vegetation, pond-grubs, and the like are due to the intense local strains in the vicinity of a pulsating bubble. The local mechanical strain is increased at least 14,700 times by the presence

* Communicated by the Author.

of any pulsating bubble smaller than the resonant size. Objects in contact with a bubble may be strained beyond the elastic limit and damaged.

WHEN intense sound-waves are maintained in water some of the dissolved gases are released. Soon after the sound begins small bubbles appear in the water; these move to and fro and increase in size, partly by absorption of gas from the water and partly by combination with other bubbles. At a particular stage of growth the bubbles pass through a brief period of violent activity during which they rush about rapidly, appearing to the eye as streaks on account of persistence of vision. They emerge from this brief interlude increased in size and relatively quiescent. Further combinations with other bubbles occur, and, finally, as the buoyancy increases, the large bubble, perhaps 1 or 2 millimetres in diameter, rises to the surface and escapes to the atmosphere. These phenomena are best observed within a system of standing waves, because in a simple progressive wave the bubbles are continually driven away from the source of sound and are difficult to observe.

Since the bubbles are small in comparison with the wave-length the alternating mechanical pressure in the sound-wave acts approximately uniformly over their surfaces; they pulsate in the simplest radial mode. Each bubble has restoring force, inertia, and damping, and pulsates as a resonant mechanical system having one degree of freedom.

The restoring force per unit area arising from the elasticity of the gas is easily shown to be

$$\gamma = \frac{3\kappa}{a} \cdot \left(p_0 + \frac{2T}{a} \right) \\ \doteq \frac{3\kappa}{a} p_0 \text{ if } \frac{2T}{a} \ll p_0, \quad (1)$$

where a is the radius of the bubble, p_0 is the hydrostatic pressure, T is the surface tension of the water-gas interface, and κ is the ratio of the specific heats of the gas at constant pressure and constant volume. Adiabatic compression and expansion have been assumed.

The effective mass α per unit area arises from the

kinetic energy of the water moved radially by the pulsation of the bubble ⁽¹⁾. It is given by

$$\alpha = \rho a / (1 + k^2 a^2), \quad . \quad . \quad . \quad . \quad (2)$$

where ρ is the density of water, $k = 2\pi/\lambda$, λ being the wave-length. If $ka \ll 1$, this becomes

$$\alpha = \rho a. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The radiation resistance or damping factor β per unit area arises from the scattering of sound energy by the bubble ⁽²⁾. It is given by

$$\beta = \rho c k^2 a^2 / (1 + k^2 a^2), \quad . \quad . \quad . \quad . \quad (4)$$

$$= \rho c k^2 a^2 \text{ if } ka \ll 1, \quad . \quad . \quad . \quad . \quad (5)$$

where c is the velocity of sound in the water.

If p is the instantaneous alternating pressure in the water in the absence of the bubble the equation of motion of the latter is

$$p = \alpha \ddot{\xi} + \beta \dot{\xi} + \gamma \xi, \quad . \quad . \quad . \quad . \quad (6)$$

where ξ is the displacement of the surface of the bubble from its equilibrium position and the dot denotes differentiation with respect to time. Resonance occurs when

$$\alpha \ddot{\xi} + \gamma \xi = 0,$$

that is, at an angular velocity ω , given by

$$\omega^2 = \gamma/\alpha, \quad . \quad . \quad . \quad . \quad (7)$$

when the equation of motion reduces to

$$p = \beta \dot{\xi}. \quad . \quad . \quad . \quad . \quad (8)$$

Using the values of α and γ given in (1) and (2),

$$\omega^2 = \frac{1}{a^2} \cdot \frac{3\kappa}{\rho} \left(p_0 + \frac{2T}{a} \right). \quad . \quad . \quad . \quad (9)$$

Since p_0 is of the order of 10^6 , and T is of the order 70, the term $2T/a$ is too small to be appreciable except for bubbles of microscopic size. Neglecting this term the formula for the resonance pulsation of a spherical bubble becomes

$$\omega = \frac{1}{a} \cdot \sqrt{\frac{3\kappa p_0}{\rho}}. \quad . \quad . \quad . \quad (10)$$

The ratio of mass reactance per unit area, $\omega\alpha$, to the radiation resistance per unit area, β , is a measure of the degree of resonance. From (3) and (5) it appears that

$$\omega\alpha/\beta = 1/ka. \quad . \quad . \quad . \quad (11)$$

Substituting the numerical values given in the table below, this ratio comes out at 70, indicating that the bubble is sharply resonant. If, however, the amplitude of vibration at resonance is calculated from (3), using a reasonable value for p , an impossibly large value is obtained; it seems that sources of damping other than radiation resistance, for example, turbulence and irreversible heat exchanges between the bubble and the water, exist.

The following table gives the radius of the resonant bubble in water for a range of frequencies; p_0 has been taken as 10^6 , k as 1.41, and ρ as 1:—

Frequency .. $\omega/2\pi$	$10^3 \sim$	$10^4 \sim$	$10^5 \sim$	$10^6 \sim$
Radius..... a	3.3 mm.	0.33 mm.	0.033 mm.	0.0038 mm.

It may be that the period of violent activity in the life-history of a bubble, mentioned above, occurs at the stage in its growth when its radius is appropriate for resonance.

In this connexion it is interesting to note that the musical sounds of running water have been attributed to the shock excitation of resonant air-bubbles ⁽³⁾.

The radial pulsation of the bubbles is possibly a cause of the destruction effects of intense sound. It is well known that vegetable matter is partially disintegrated by intense sound ⁽¹⁾; small pond-grubs are killed and ultimately torn to shreds. These objects probably serve as nuclei for the development of bubbles and thereby bring about their own destruction. At the point of contact there must be a tearing action resulting from the radial pulsation of the bubble. The theory throws some light on the point. For bubbles small in comparison with the resonant bubble

$$p = \gamma \xi = \frac{3\kappa}{a} p_0 \xi.$$

The tangential strain in the gas at the surface is given by

$$\xi/a = p/3\kappa p_0. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The water at the bubble surface yields to this strain by flowing; but if any solid matter is near or in contact with the surface it may not be able to endure the

repeated deformation and so be mechanically damaged and perhaps torn. Two cases occur: in the first the solid matter is inadequate in cross-section and strength to resist the stresses imposed by the pulsation of the bubble and is damaged; in the second the solid matter resists the stresses and modifies the pulsation of the bubble. In the absence of a bubble the strain is $p/3K$, where K is the bulk modulus of the liquid; the presence of the bubble gives rise to the much larger strain $p/3\kappa p_0$. The ratio of increase is $K/\kappa p_0$; if we assume $K=2.06 \times 10^{10}$ dynes per sq. cm., $p_0=10^6$ dynes per sq. cm., and $\kappa=1.4$, this ratio is about 14,700. The increase is the same for all bubbles appreciably smaller than the resonant bubble and for all frequencies. All bubbles, including those too small to see with the naked eye, are therefore likely to contribute to the destructive effect.

It may easily be shown in the same way that the resonant bubble and also larger bubbles are able to produce greatly enhanced strains in solid objects; they are not nearly so numerous, however, and their total effect is probably relatively unimportant.

The existence of the resonant bubble was first suspected by Mr. B. S. Smith, who suggested the problem to me. My thanks are due to him and also to Mr. S. Butterworth and Dr. A. B. Wood for their interest and suggestions, and to the Admiralty for permission to publish this paper.

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20th November, 1934.

XCIX. *Notices respecting New Books.*

A Study of Crystal Structure and its Applications. By W. P. DAVEY. [Pp. 695+xi.] (McGraw-Hill Publishing Co., Ltd. Price 45s.)

THIS book is one of the latest additions to the publishers' well known International Series in Physics, and has been produced in the same excellent style as the other books of the series. The author's aim has been to write a book "for those who have the maturity and training to be expected of a college graduate," and its purpose is "to put such a person in possession of enough knowledge of the theory and technique of crystal analysis to enable him to read the literature intelligently and to do independent experimental work," but it is not intended to be "a summary of the data to be found in the literature." There is, however, considerable overlapping between certain parts of this book and other well-known books on crystal analysis, as we shall point out.

A more serious criticism is that in several ways the book directs too much attention to lines of thought that are now largely out of date. This applies particularly to the treatment of X ray intensities; on pp. 82, 85, 95, 144, 287, and elsewhere, the amplitude diffracted by an atom is taken as proportional to the atomic number (though the scattering factor is mentioned in foot-notes), and with this assumption formulæ are obtained for reflected intensities which are then corrected empirically for the so-called "normal decline of intensity" with increasing angle of diffraction. This method was, perhaps, good enough ten years ago when little was known quantitatively about how atoms diffract X-rays, but is no longer necessary. Had the author introduced the idea of the scattering factor earlier in the text, and applied it in the various equations connected with X-ray intensities, he would have had no necessity to resurrect an obsolete method; the reviewer cannot recall having seen a single analysis since 1930 involving this empirical method.

The first part of the book is briefly as follows:—

After preliminary chapters on X-ray diffraction and the theory of crystal lattices, four chapters describe the Laue, Bragg, powder, and rotating crystal methods of analysis. The treatment of these chapters is particularly lucid and, though it does not present any novel features, should be very useful to the beginner. The text could have been reduced to some extent by referring the reader more frequently to other books in which these topics are discussed. Following this, the theory of space groups is treated in detail.

The structure factor is dealt with in Chapter X. along much the same lines as in Compton's 'X-Rays and Electrons.' If the author's aim was not to repeat work easily accessible elsewhere, one wonders what justification there is for much of this chapter. Both here and in other parts of the book (*cf.* pp. 61 to 67, 178) space is wasted on elementary mathematics, such as simple trigonometrical expansions, statements that the sine of a small angle is equal to the angle in radians and that $\cos^2 \theta = 1 - \sin^2 \theta$. There is little reference to the work done on scattering factors during the last five years, and no references are given to the extensive lists of scattering factors now almost universally used in crystal analysis. In dealing with the application of Fourier analysis, no mention is made of double Fourier analysis, which is being more and more widely used especially in connexion with organic crystals. Chapter XI. deals with applications of the structure factor method and gives in fairly full detail a number of typical analyses.

Although the book is essentially a practical book, it omits a lot of data often required in the laboratory. For example, there are no tables of absorption coefficients. Voltage measurements by the spark-gap method are mentioned, but no figures are given. The photographic effect of X-rays and the connexion between blackness and intensity is very important but is not discussed. Some wave-length tables would also be useful. The reviewer personally would have preferred to have these data ready to hand rather than the extensive lists of space groups which occupy as many as fifty nine pages, and which can easily be found elsewhere.

The second part of the book is concerned with physical and chemical applications, and on the average is better done and more original than the first part. Chapter XIII. deals with atomic and ionic sizes in crystals, but rather too much attention is given to some of the earlier work on this subject and later developments are dismissed too briefly. It is very pleasing to find chapters on crystal growth (Chapter XII.) and on the refraction of light by crystals (Chapter XIV.), for these are aspects of the subject which are receiving considerable attention at the present time and which have not been summarized elsewhere. Chapter XV. on lattice energies and the Born theory of ionic crystals follows the older methods; the newer developments of Born's theory, first published in 1932, and the part played by the van der Waal forces of attraction are not mentioned. This is a sad omission, for though the van der Waal forces are weak compared with the Coulomb forces, they appear to play an important part in determining the most stable type of structure for certain crystals. Chapter XVI. summarizes very well the diffraction of X-rays

by amorphous materials (glasses, gases, etc.). Short Chapters deal with crystal orientation and solid solutions, and Chapter XIX. summarizes a large amount of chemical information which has been obtained from crystal analysis, particularly in connexion with organic compounds.

The most regrettable feature about the book as a whole is that it is unnecessarily long; a vigorous pruning would produce a shorter, better, and less expensive volume. Too much elementary matter, mathematical and otherwise, is included, and there is too much over-lapping with other books; this applies particularly to the first half of the book. The second half is good, but would have been better if more attention had been given to developments during the last few years, such as the later Born theory, the theory of the homopolar bond and of directed valence, Pauling's work on the balancing of electrostatic valencies, the distribution of electrons in atoms and ions in relation to the structure factor. Nevertheless, apart from these defects, the book may be very useful to the beginner in crystal analysis, who will doubtless be more concerned with practical difficulties than with the latest theoretical developments.

General Astronomy. By H. SPENCER JONES. Second Edition. [Pp. viii+437.] (Arnold: London, 1934. Price 12s. 6d.)

ASTRONOMY, although it is superficially one of the least "useful" of the sciences, has perhaps the widest diversity of appeal of all of them. The number of professional astronomers is comparatively small, but the subject is one which is of fundamental interest to the mathematician and the physicist; the amateur astronomer has an inexhaustible field in which he can still make significant discoveries, unaided by costly equipment, and many members of the non-scientific public find an irresistible fascination in astronomy, while other sciences leave them cold. For the student and the amateur to whom it is primarily addressed, the Astronomer Royal's book is probably the most satisfying of the works on general astronomy at present available. For the specialist in particular branches of astronomy also it will be of value owing to its comprehensiveness; and while it is by no means "popular," it is written in such a way that the reader with a minimum of technical knowledge will be able to gain an understanding of methods and principles as well as detailed information about results and conclusions.

The first edition was published in 1922. The new edition is revised, considerably enlarged, and brought thoroughly up to date. The book opens—perhaps rather discouragingly

for the non-professional—with definitions and explanations of the various technical terms associated with the specification of stellar positions. Then follow chapters on the Earth, the Earth in relation to the Sun, and astronomical instruments and observations. After this the various heavenly bodies are considered in turn—the Moon, the Sun, the planets, comets and meteors, stars and nebulae. The last two chapters deal with extra-galactic systems, and with stellar constitution and evolution.

Throughout, the presentation is straightforward and clear. It is essentially unpretentious; the facts are allowed to speak for themselves, there being no superfluous comment. The amount of information given is remarkable; nothing of importance seems to have escaped notice. Among recent work on planets, of which excellent accounts are included, may be mentioned that on Pluto, on the two new asteroids of 1932, one of which approaches to within three million miles of the Earth's orbit, and on the atmospheres of planets, studied spectroscopically, and by photography in light of different colours; on stars, on novae, on cepheid variables, and on stellar spectra generally.

The almost complete absence of references is surprising. To the reviewer it seems that the value of a book of this kind would have been greatly enhanced by a short bibliography to each chapter; but the book is written almost as though there were no other astronomical text-books in existence. Numerous references to original papers are hardly required, but a few such references would have been useful in some places to supplement the author's very objective presentation of rival theories. As a minor criticism it may be suggested that the significance of many of the numerical data which are incorporated in the text, as in the account of planets, making for somewhat trying reading, might have been more readily appreciated if they had been set out in tabular form.

As a straightforward account of general astronomy, the book could hardly be bettered. The book is admirably produced, and containing twenty-seven excellent plates, is a marvel of cheapness. It is a book which should be bought by all those who are seriously interested in astronomy.

Rayleigh's Principle. By G. TEMPLE and W. G. BICKLEY. [Pp. ix+156.] (Oxford University Press, 1933. Price 14s. 0d. net.)

THIS book puts forward and justifies a rapid means of calculating the fundamental period of a vibrating system or the condition of stability of an elastic system with the degree of accuracy usually demanded in engineering problems. The

utility of 'Rayleigh's Principle' for such determinations has been realized in recent years, but the approximate values so obtained are in excess of the true values, although they are well within the engineering degree of tolerance. The authors develop Rayleigh's energy method so as to provide both upper and lower estimates of the true value required with close and known degrees of approximation. Numerous applications are discussed, and exercises are suggested for the consideration of the reader.

Ions, Electrons, and Ionising Radiations. By J. A. CROWTHER.
Sixth Edition. (London: Edward Arnold & Co. Price 12s. 6d. net.)

It is not often that a text-book can be welcomed with such unreserved pleasure as this new edition of Professor Crowther's now famous introduction to Modern Physics. The last few years have brought an embarrassment of riches from the research laboratories. Amidst all this new wealth, this lucid writer has, as we should expect, kept his head and his balance, and the result will undoubtedly help those of his readers to whom the subject is fresh to keep theirs.

It is difficult to imagine a better treatment; it is quite certain that none exists.

Of the two new chapters, one on the Nucleus and one on Neutrons, Positrons, and Cosmic Radiation, it need only be said that they are worthy of their elder brothers. Although the text has been drastically revised, it is wise to have left the descriptions of the earlier work to stand as they were first written. For they do indeed—as the author hopes—preserve the freshness of contemporary accounts. The present reviewer, who was fortunate enough as a student to have the privilege of reading the first edition in its manuscript form, can still remember the thrill that gave him and the enthusiasm for the living science it inspired. But enthusiasm is not enough; Professor Crowther knows this and does not shirk critical discussion where it is possible. Nor does he, although the mathematical treatment is simple enough for a second year Honours man, give many hostages to easy evasions.

In a fairly careful reading, only two minor errors have been detected in fig. 86, and in the somewhat confusing existence of two quite different equations 131. But we should like to see Mahomet's coffin, which has done such good service in the cloud chamber, given at last a decent burial.

Publishers and printers have collaborated to produce a volume that is eminently reasonable in price and a pleasure to eye and hand.

Crystals and the Polarizing Microscope: a Handbook for Chemists and Others. By N. H. HARTSHORNE, Ph.D., M.Sc., and A. STUART, M.Sc. [Pp. viii+272.] (London: Edward Arnold & Co., 1934. Price 16s. net.)

THIS well-written book brings together the essential parts of crystal morphology and crystal optics and applies them to the theory and use of the polarizing microscope. Convincing demonstration is then given of the value of this instrument in many phases of chemical work.

An excellent introduction to crystallography is presented in Chapters 1 and 2, which deal with crystal symmetry and external form. The internal structure of crystals is omitted, and the reader is referred for such information to standard works on X-ray crystallography. The stereographic projection method might have been added to this section, since the usual crystallographic description of the external crystal form is recommended, and possibly, too, some explanation of the reason for using four axes to describe hexagonal and trigonal crystals. Considerable space has been devoted to Chapter 3, which sets forth the fundamentals of crystal optics and deals clearly with special cases, relevant to the later sections, by means of the Huyghen constructions. The remaining chapters give fullest details of design and use of the polarizing microscope and of the methods of attack on various problems in the laboratory and in industry.

The most general application lies in the identification of a crystal by means of its optical properties, and the avoidance thereby of much laborious analytical work. This is shown to be particularly useful in studying heterogeneous equilibria and aqueous systems; further, in synthesizing compounds much time is saved by examining the crude reaction mixture for the presence of the desired compound. Some account of the application of the polarizing microscope to the study of fibre structures would have been welcome here, particularly in view of the rather inadequate treatment presented in books on textile researches.

Throughout the book there is an abundance of practical detail, and in many instances the diagrams not only illustrate the text, but also provide useful guides to actual practical work. The authors are to be congratulated upon this attractive and useful book, which will be appreciated by many workers engaged upon physical or chemical research.

[The Editors do not hold themselves responsible for the views expressed by their correspondents.]

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END OF THE NINETEENTH VOLUME.

FIG. 1.

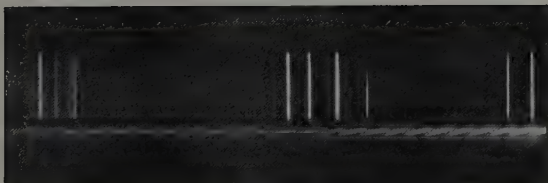
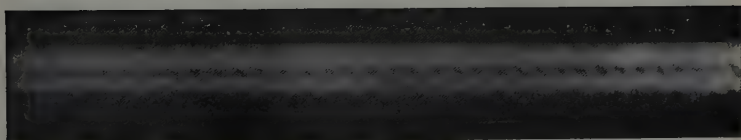


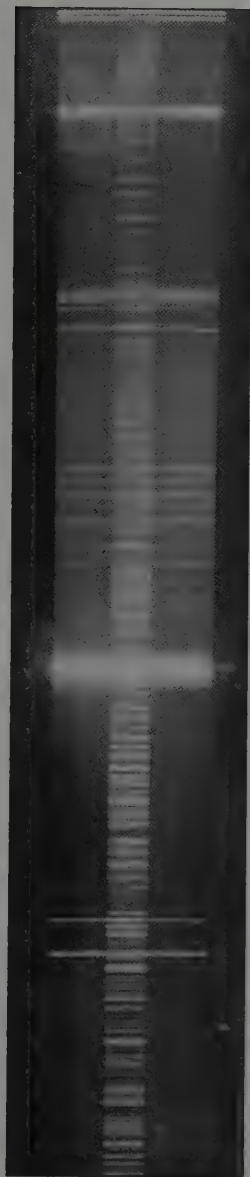
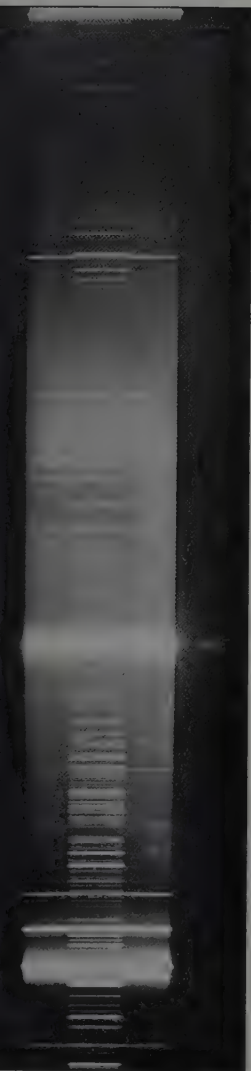
FIG. 2.



(b)

(c)

(a)



(a) Tetralin Quinine sulphate filter (b) Tetralin. *m*-dinitro benzene in benzene filter. (c) Tetralin



a. Formamide. *b.* Benzophenone. *c.* Acetanilide.

Wood Reserve

Wool Research